TEORIJA POVRŠINSKIH NOSAČA

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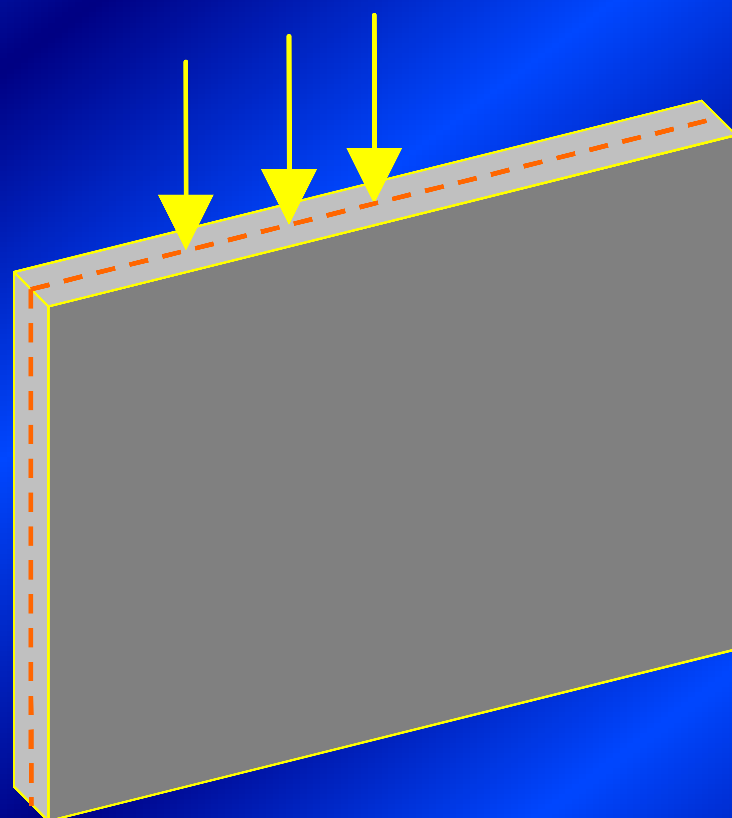
Klasifikacija nosača
(topologija, naponsko-deformacijsko stanje i funkcija)

- Linijski nosači (1D, zatege, grede, stubovi...)
- Površinski nosači (2D, ploče, zidovi, membrane, ljuske...)
- Prostorni nosači (3D, blokovi, jezgra, brane...)

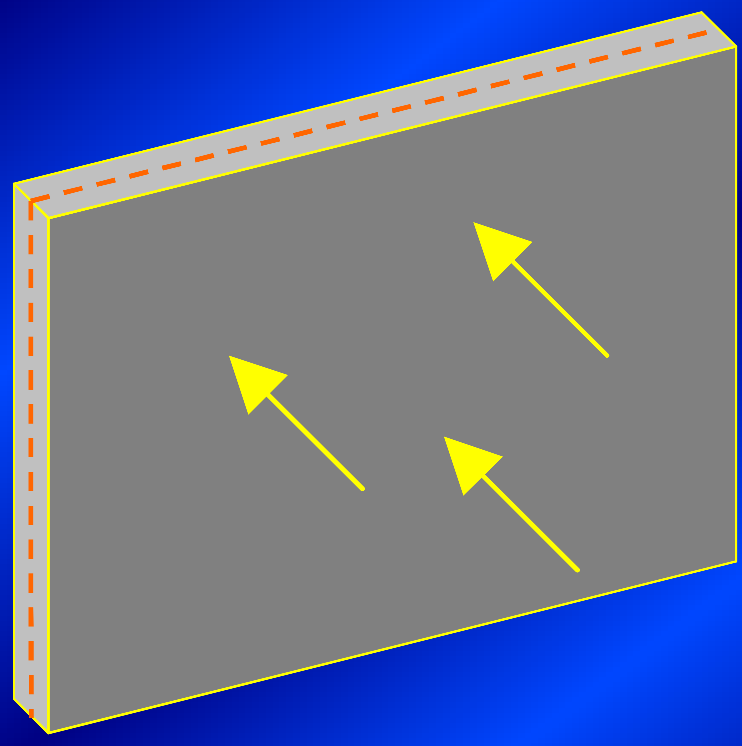
Površinski nosači

- Zidovi (ravne površi, opterećenje u sopstvenoj ravni, aksijalna, smičuća i fleksiona krutost u sopstvenoj ravni: aksijalno naprezanje, smicanje i savijanje)
- Ploče (ravne površi, opterećenje normalno na sopstvenu ravan, smičuća, fleksiona i torziona krutost u normalnim ravnima: savijanje, smicanje, torzija)
- Membrane (krive površi, aksijalna krutost u tangentnoj ravni, aksijalno naprezanje)
- Ljuske (krive površi, aksijalna krutost u tangentnoj ravni, smičuća i fleksiona krutost u normalnim ravnima: aksijalno naprezanje savijanje, smicanje, torzija)

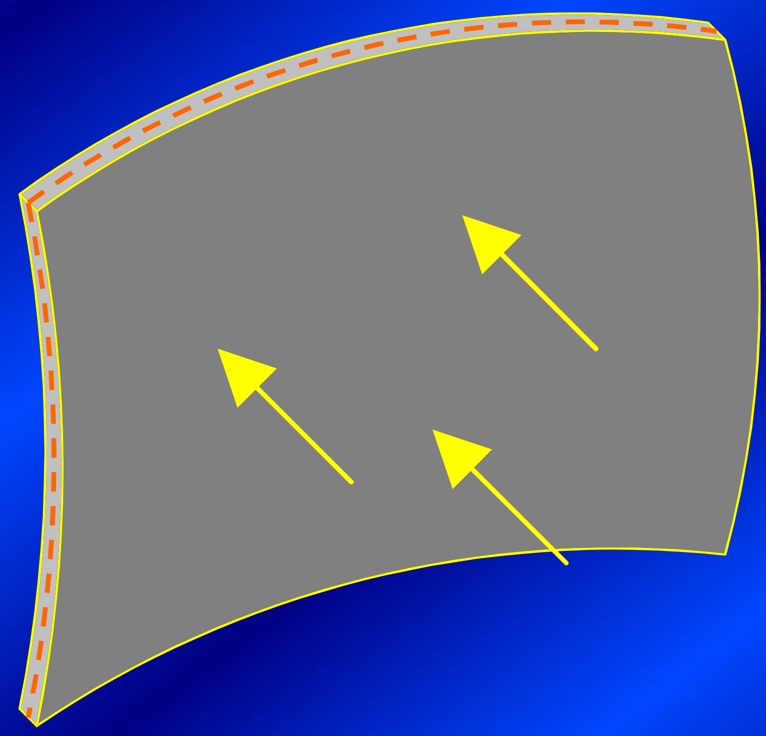
Zidovi (ravne površi, poligonalna ili kriva kontura)



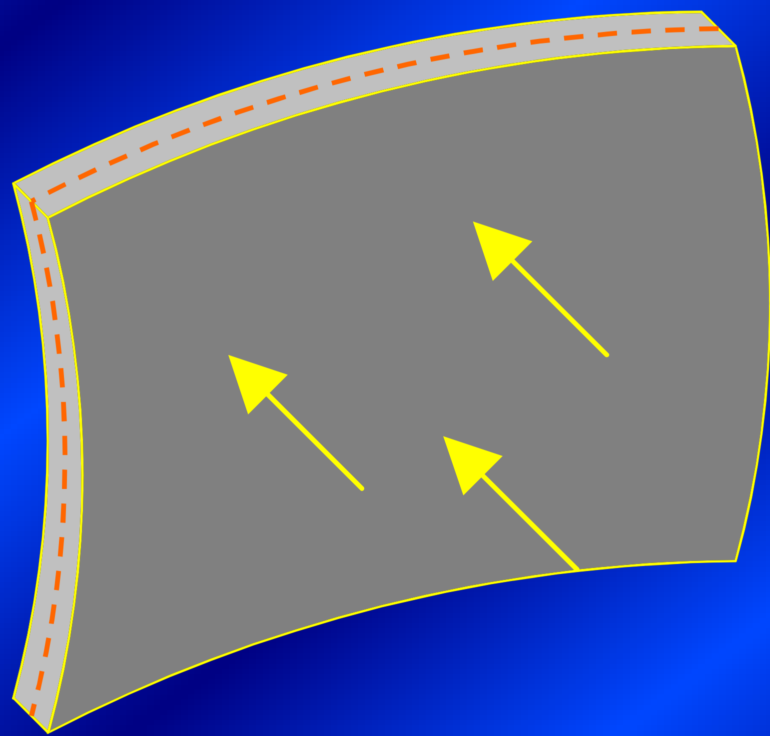
Ploče (ravne površi, poligonalna ili kriva kontura)



Membrane (cilindrične ili sferne površi)



Ljuske (cilindrične ili sferne površi)

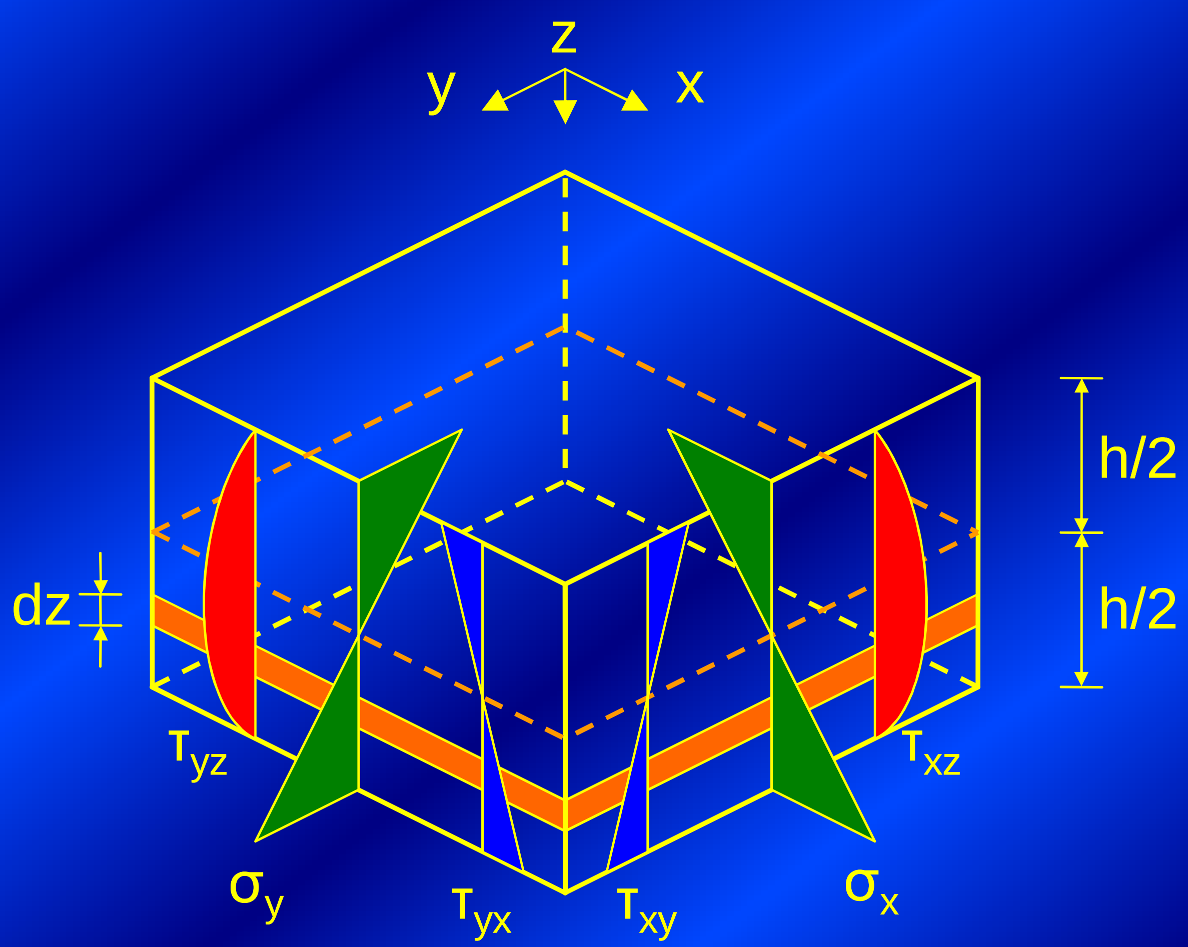


PLOČE OPTEREĆENE NA SAVIJANJE

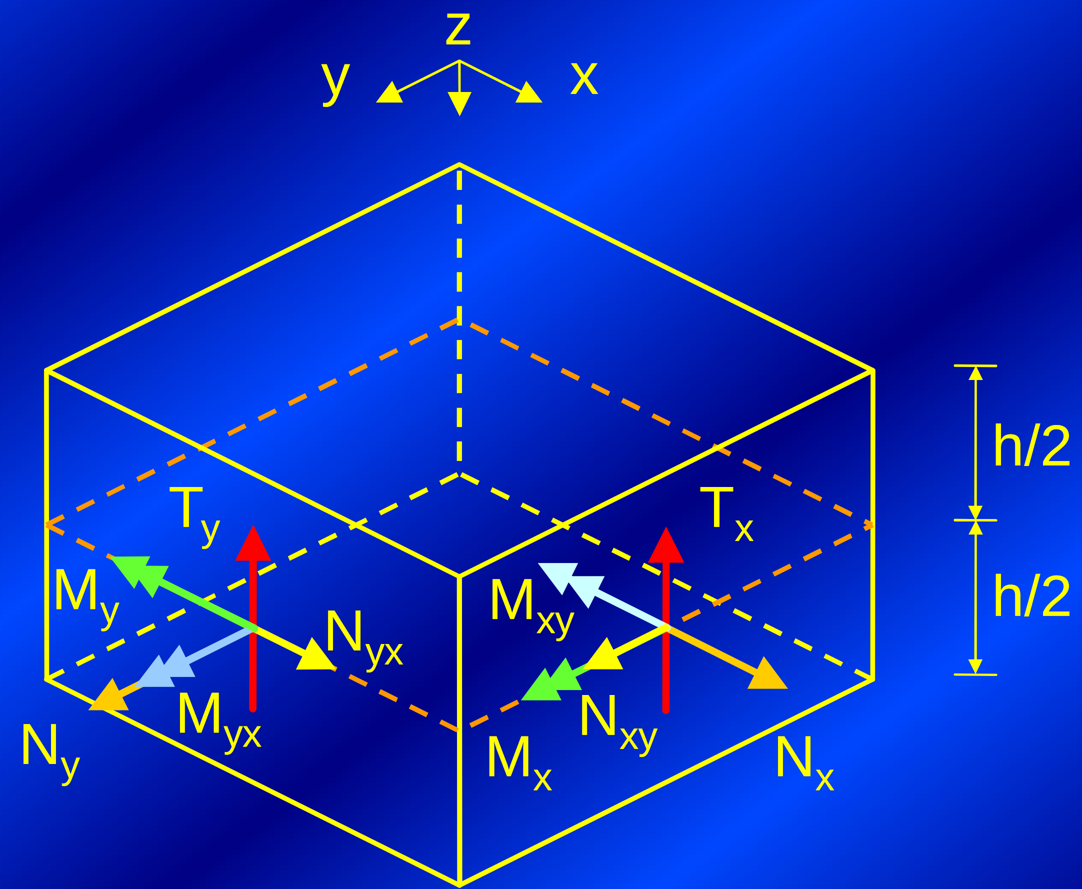
Osnovni pojmovi

- Ploče - tela ograničena dvema (paralelnim) ravnima i cilindričnom površinom ortogonalnom na njih
- Debljina ploče - "h" - rastojanje (paralelnih) ravni
- Srednja ravan ploče - ravan koja polovi debljinu ploče - pod opterećenjem prelazi u elastičnu površinu
- Kontura ploče - kriva preseka srednje ravni i cilindrične površine koja ograničava ploču

Komponentalni naponi



Sile u presecima



Sile u presecima i komponentalni naponi (savijanje)

$$M_x = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_x \cdot z \cdot dz \qquad M_y = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_y \cdot z \cdot dz$$

$$T_{xz} = T_x = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \tau_{xz} \cdot dz \qquad T_{yz} = T_y = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \tau_{yz} \cdot dz$$

$$M_{xy} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \tau_{xy} \cdot z \cdot dz \qquad M_{yx} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \tau_{yx} \cdot z \cdot dz$$

Klasifikacija ploča prema debljini i ponašanju pod opterećenjem

- membrane (h/b<1/8-1/100 - mala fleksiona krutost i veliki uticaj aksijalnih deformacija)...
- tanke ploče (1/8-1/100<h/b<1/5-1/8 - srednja fleksiona krutost)...
- debele ploče (h/b>1/5-1/8 - velika fleksiona krutost i veliki uticaj smicanja)...

Pretpostavke teorije tankih ploča (Kirchoff, 1850)

- pravolinijski element koji je normalan na srednju ravan ploče, posle deformacije ostaje prav, normalan na srednju ravan i ne menja dužinu...
- posle deformacije se ne menja dužina i ugao između linijskih elemenata srednje ravni ploče...
- normalni "σ_z" naponi u ravnima paralelnim srednjoj ravni mogu da se zanemare u odnosu na ostale komponentalne napone...

Komponentalne deformacije ploče

- na osnovu prve pretpostavke teorije tankih ploča:

$$\varepsilon_z = \frac{\partial w}{\partial z} \qquad w = w(x, y)$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0 \qquad \Rightarrow \qquad \frac{\partial u}{\partial z} = -\frac{\partial w}{\partial x}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0 \qquad \Rightarrow \qquad \frac{\partial v}{\partial z} = -\frac{\partial w}{\partial y}$$

- integrisanjem po "z", (za z=0, u=v=0):

$$u = -z \cdot \frac{\partial w}{\partial x} \qquad v = -z \cdot \frac{\partial w}{\partial y}$$

$$\varepsilon_x = \frac{\partial u}{\partial x} = -z \cdot \frac{\partial^2 w}{\partial x^2}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = -z \cdot \frac{\partial^2 w}{\partial y^2}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z \cdot \frac{\partial^2 w}{\partial x \cdot \partial y}$$

Komponentalni naponi

$$\varepsilon_x = \frac{1}{E}\cdot(\sigma_x - \nu\cdot\sigma_y) \qquad \varepsilon_y = \frac{1}{E}\cdot(\sigma_y - \nu\cdot\sigma_x) \qquad \gamma_{xy} = \frac{1}{G}\cdot\tau_{xy}$$

$$\sigma_x = \frac{E}{1-\nu^2}\cdot(\varepsilon_x + \nu\cdot\varepsilon_y) \qquad \sigma_y = \frac{E}{1-\nu^2}\cdot(\varepsilon_y + \nu\cdot\varepsilon_x) \qquad \tau_{xy} = G\cdot\gamma_{xy}$$

$$\sigma_x = \frac{E}{1-\nu^2}\cdot z\cdot\left(\frac{\partial^2 w}{\partial x^2} + \nu\cdot\frac{\partial^2 w}{\partial y^2}\right) \qquad \sigma_y = \frac{E}{1-\nu^2}\cdot z\cdot\left(\frac{\partial^2 w}{\partial y^2} + \nu\cdot\frac{\partial^2 w}{\partial x^2}\right)$$

$$\tau_{xy} = G\cdot z\cdot\frac{\partial^2 w}{\partial x\cdot\partial y}$$

Sile u presecima

$$M_x = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_x \cdot z \cdot dz = -\frac{E}{1-v^2} \cdot \left(\frac{\partial^2 w}{\partial x^2} + v \cdot \frac{\partial^2 w}{\partial y^2}\right) \cdot \int_{-\frac{h}{2}}^{-\frac{h}{2}} \sigma_x \cdot z^2 \cdot dz \Rightarrow$$

$$M_x = -\frac{E \cdot h^3}{12(1-v^2)} \cdot \left[\frac{\partial^2 w}{\partial x^2} + v \cdot \frac{\partial^2 w}{\partial y^2}\right]$$

$$M_y = -\frac{E \cdot h^3}{12(1-v^2)} \cdot \left[\frac{\partial^2 w}{\partial y^2} + v \cdot \frac{\partial^2 w}{\partial x^2}\right]$$

$$M_{xy} = -\frac{E \cdot h^3}{12(1-v^2)} \cdot (1-v) \cdot \frac{\partial^2 w}{\partial x \cdot \partial y}$$

Ekstemne vrednosti sila u presecima

$$M_{1,2} = \frac{M_x + M_y}{2} \pm \frac{1}{2} \cdot \sqrt{(M_x - M_y)^2 + 4M_{xy}^2} \qquad tg2\alpha_{1,2} = \frac{2 \cdot M_{xy}}{M_x - M_y}$$

$$M_{t,1,2} = \pm \frac{1}{2} \cdot \sqrt{(M_x - M_y)^2 + 4M_{xy}^2}$$

$$\sigma_{x,max} = \pm \frac{6M_x}{h^2} \qquad \sigma_{y,max} = \pm \frac{6M_y}{h^2} \qquad \tau_{xy,max} = \pm \frac{6M_{xy}}{h^2} \qquad (\text{za } z = \pm \frac{h}{2})$$

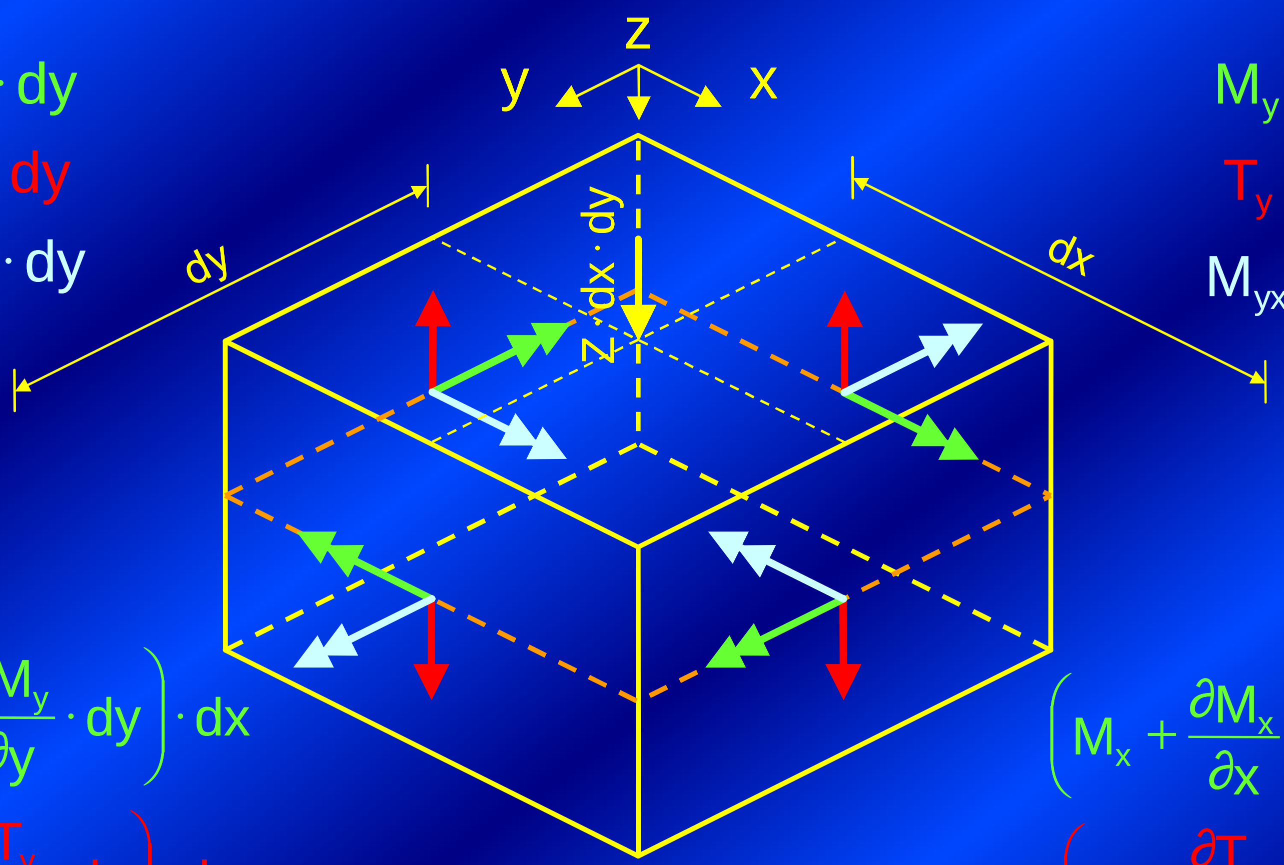
$$\tau_{xz,max} = \frac{3T_x}{2h} \qquad \tau_{yz,max} = \frac{3T_y}{2h} \qquad (\text{za } z = 0)$$

Uslovi ravnoteže

$M_x \cdot dy$

$T_x \cdot dy$

$M_{xy} \cdot dy$



$M_y \cdot dx$

$T_y \cdot dx$

$M_{yx} \cdot dx$

$$\left(M_y + \frac{\partial M_y}{\partial y} \cdot dy\right) \cdot dx$$

$$\left(T_y + \frac{\partial T_y}{\partial y} \cdot dy\right) \cdot dx$$

$$\left(M_{yx} + \frac{\partial M_{yx}}{\partial y} \cdot dy\right) \cdot dx$$

$$\left(M_x + \frac{\partial M_x}{\partial x} \cdot dx\right) \cdot dy$$

$$\left(T_x + \frac{\partial T_x}{\partial x} \cdot dx\right) \cdot dy$$

$$\left(M_{xy} + \frac{\partial M_{xy}}{\partial x} \cdot dx\right) \cdot dy$$

$$\sum X \equiv 0 \qquad \sum Y \equiv 0 \qquad \sum M_z = 0$$

$$\sum M_y = 0$$

$$\Downarrow$$

$$\frac{\partial M_x}{\partial x} \cdot dx \cdot dy + \frac{\partial M_{xy}}{\partial y} \cdot dx \cdot dy - T_x \cdot dx \cdot dy - \frac{\partial T_x}{\partial x} \cdot dx \cdot dy \cdot \frac{dx}{2} = 0$$

$$\Downarrow$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - T_x = 0$$

$$\sum M_x = 0 \qquad \sum Z = 0$$

$$\Downarrow \qquad \Downarrow$$

$$\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - T_y = 0 \qquad \frac{\partial T_x}{\partial x} + \frac{\partial T_y}{\partial y} + Z = 0$$

$$T_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} = -\frac{E \cdot h^3}{12(1-v^2)} \cdot \left[\frac{\partial^3 w}{\partial x^3} + v \cdot \frac{\partial^3 w}{\partial x \cdot \partial y^2} \right]$$

$$T_y = \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} = -\frac{E \cdot h^3}{12(1-v^2)} \cdot \left[\frac{\partial^3 w}{\partial y^3} + v \cdot \frac{\partial^3 w}{\partial x^2 \cdot \partial y} \right]$$

$$\frac{\partial T_x}{\partial x} + \frac{\partial T_y}{\partial y} + Z = 0$$

$$k = \frac{E \cdot h^3}{12(1-v^2)}$$

Parcijalna diferencijalna jednačina savijanja pravougaone ploče

$$\frac{\partial T_x}{\partial x} + \frac{\partial T_y}{\partial y} + Z = 0 \qquad \frac{\partial^2 M_x}{\partial x^2} + 2 \cdot \frac{\partial M_{xy}}{\partial x \cdot \partial y} + \frac{\partial^2 M_y}{\partial y^2} + Z = 0$$

$$\frac{\partial^4 w}{\partial x^4} + 2 \cdot \frac{\partial^4 w}{\partial x^2 \cdot \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{Z}{k}$$

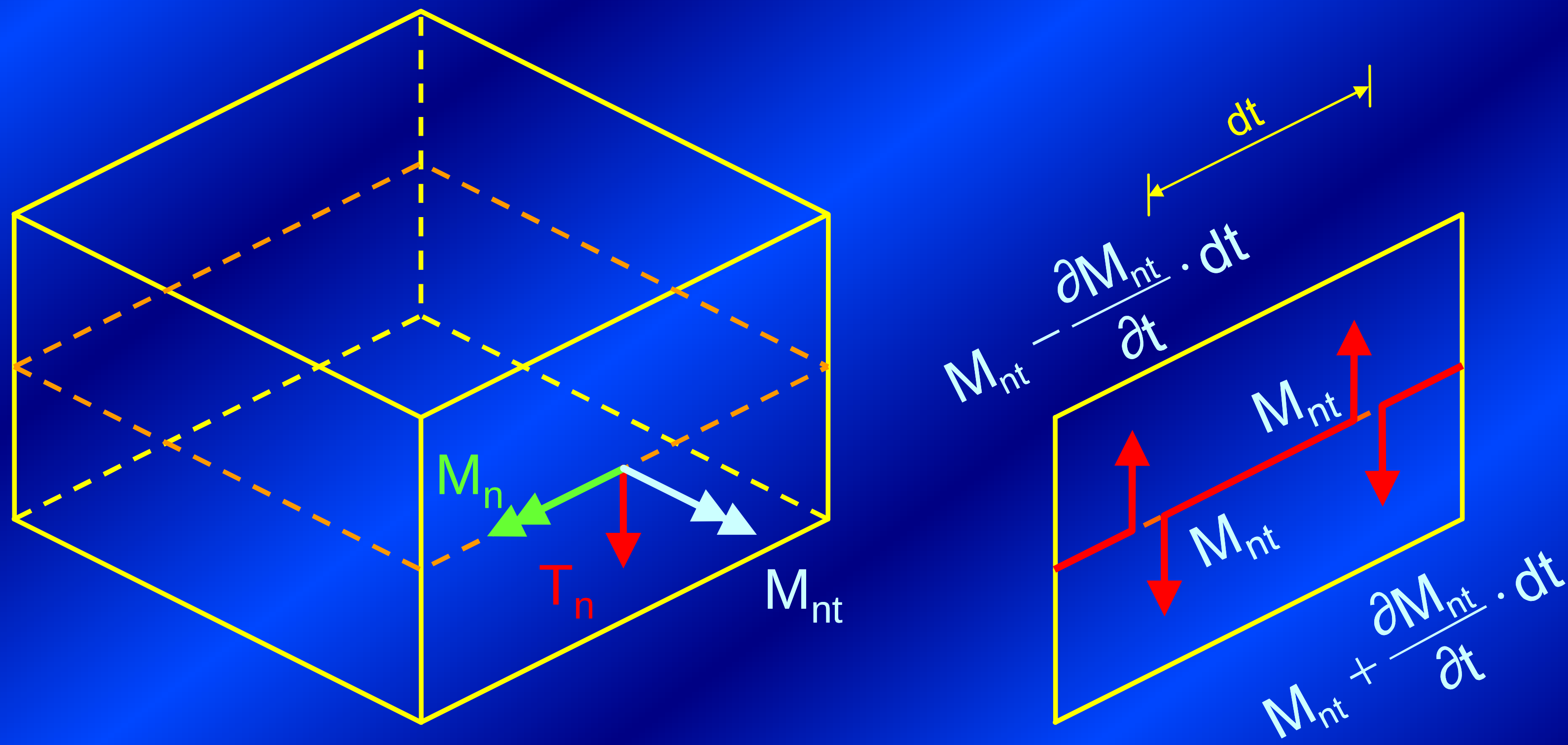
$$\Delta w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \qquad \Delta\Delta w = \frac{Z}{k}$$

$$M = -k \cdot \Delta w \qquad \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{M}{k} \qquad \frac{\partial^2 M}{\partial x^2} + \frac{\partial^2 M}{\partial y^2} = -Z$$

Konturni uslovi savijanja ploče

- statički konturni uslovi - sile u presecima i/ili linijska opterećenja
- geometriski konturni uslovi - uslovi oslanjanja - ugib i/ili nagib tangente
- mešoviti konturni uslovi - kombinacija geometrijskih i statičkih konturnih uslova

Statički konturni uslovi savijanja ploče



$$\Delta M_{nt} = \frac{\partial M_{nt}}{\partial t} \cdot dt \qquad \overline{T}_n = T_n + \frac{\partial M_{nt}}{\partial t}$$

$$\left.\begin{aligned} M_x &= 0 \\ \overline{T}_x &= T_x + \frac{\partial M_{xy}}{\partial y} = 0 \end{aligned}\right\}$$

slobodna ivica
sa normalom u
pravcu x-ose

$$\left.\begin{aligned} M_y &= 0 \\ \overline{T}_y &= T_y + \frac{\partial M_{xy}}{\partial x} = 0 \end{aligned}\right\}$$

slobodna ivica
sa normalom u
pravcu y-ose

$$\overline{T}_x = T_x + \frac{\partial M_{xy}}{\partial y} = -k \cdot \left[\frac{\partial^3 w}{\partial x^3} + (2-\nu) \cdot \frac{\partial^3 w}{\partial x \cdot \partial y^2} \right]$$

$$\overline{T}_y = T_y + \frac{\partial M_{xy}}{\partial x} = -k \cdot \left[\frac{\partial^3 w}{\partial y^3} + (2-\nu) \cdot \frac{\partial^3 w}{\partial y \cdot \partial x^2} \right]$$

Geometrijski konturni uslovi savijanja ploče

$$\left.\begin{aligned} w &= 0 \\ \frac{\partial w}{\partial x} &= 0 \end{aligned}\right\}$$

uklještena ivica sa normalom u pravcu x-ose

$$\left.\begin{aligned} w &= 0 \\ \frac{\partial w}{\partial y} &= 0 \end{aligned}\right\}$$

uklještena ivica sa normalom u pravcu y-ose

$$\left.\begin{aligned} \frac{\partial w}{\partial t} &= 0 \\ \frac{\partial w}{\partial n} &= 0 \end{aligned}\right\} \Rightarrow \frac{\partial^2 w}{\partial n \cdot \partial t} = 0 \Rightarrow M_{nt} \equiv 0 \Rightarrow T_n \equiv \overline{T}_n$$

Mešoviti konturni uslovi savijanja ploče

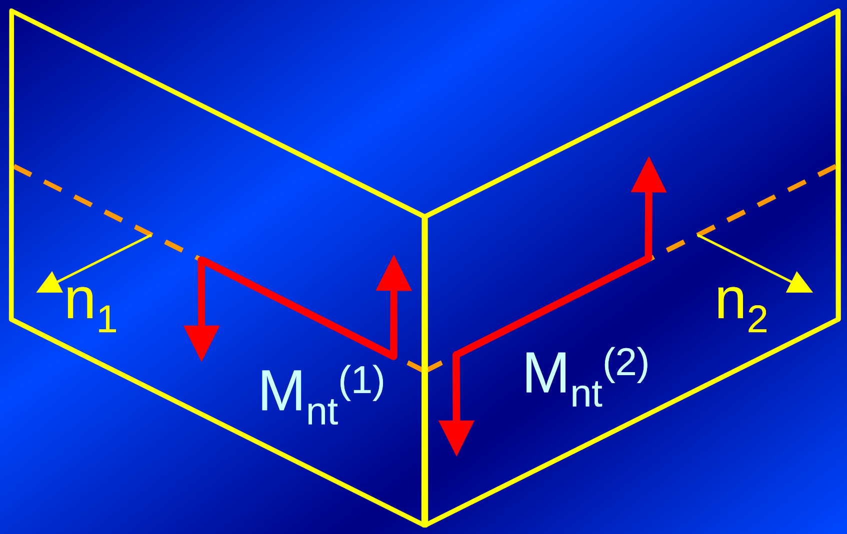
- slobodno oslonjena ivica sa normalom u pravcu x-ose

$$\begin{aligned} w &= 0 \\ M_x &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} & w = 0 \\ & \frac{\partial^2 w}{\partial x^2} + \nu \cdot \frac{\partial^2 w}{\partial y^2} = 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} & w = 0 \\ & \frac{\partial^2 w}{\partial x^2} = 0 \end{aligned}$$

- slobodno oslonjena ivica sa normalom u pravcu y-ose

$$\begin{aligned} w &= 0 \\ M_y &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} & w = 0 \\ & \frac{\partial^2 w}{\partial y^2} + \nu \cdot \frac{\partial^2 w}{\partial x^2} = 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} & w = 0 \\ & \frac{\partial^2 w}{\partial y^2} = 0 \end{aligned}$$

Sile u uglovima ploče



$$P = M_{nt}^{(2)} - M_{nt}^{(1)}$$

- dopunski konturni uslov

$$M_{nt}^{(1)} = -M_{nt}^{(2)} = M_{nt}$$

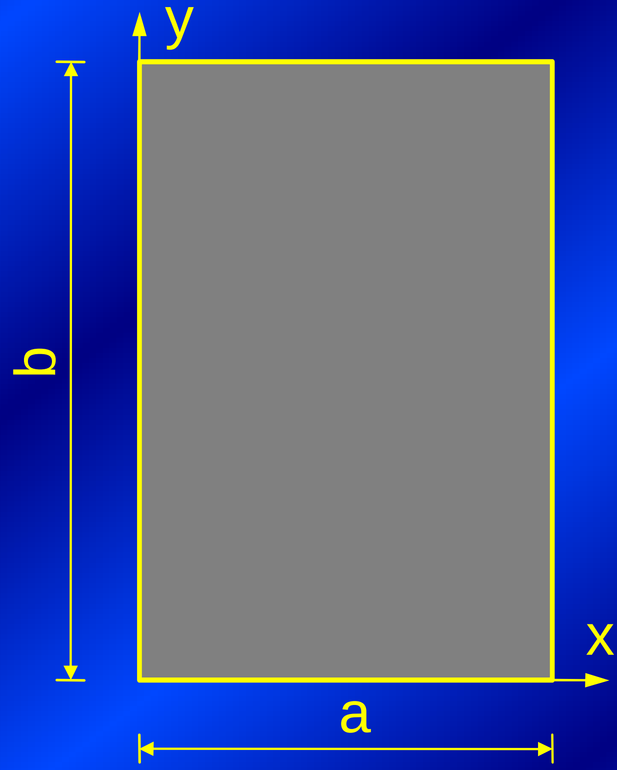
$$P = M_{nt}^{(2)} - M_{nt}^{(1)} = 0$$

$$P = 2M_{nt} = 0 \quad \Rightarrow \quad M_{nt} = 0$$

Analitičke metode za analizu i proračun ploča

- "direktno" rešavanje diferencijalne jednačine (Navier, Maurice Lévy,...)

Slobodno oslonjena pravougaona ploča (Navier, 1823)



konturni uslovi:

- x=0 i x=a

$$w = 0$$

$$\frac{\partial^2 w}{\partial x^2} = 0$$

- y=0 i y=b

$$w = 0$$

$$\frac{\partial^2 w}{\partial y^2} = 0$$

$$\Delta\Delta w = \frac{\partial^4 w}{\partial x^4} + 2 \cdot \frac{\partial^4 w}{\partial x^2 \cdot \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{Z}{k}$$

$$w(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn}$$

$$w_{mn} = A_{mn} \cdot \sin\frac{m \cdot \pi \cdot x}{a} \cdot \sin\frac{n \cdot \pi \cdot y}{b}$$

$$
w(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn} = \begin{matrix} w_{11} & + w_{12} + & \cdots & + w_{1n} + \\ + w_{21} & + w_{22} + & \cdots & + w_{2n} + \\ \vdots & \vdots & \ddots & \vdots \\ + w_{21} & + w_{21} + & \cdots & + w_{mn} \end{matrix}
$$

$$
\frac{\partial^4 w_{mn}}{\partial x^4} = A_{mn} \cdot \frac{m^4 \cdot \pi^4}{a^4} \cdot \sin\frac{m \cdot \pi \cdot x}{a} \cdot \sin\frac{n \cdot \pi \cdot y}{b}
$$

$$
\frac{\partial^4 w_{mn}}{\partial x^2 \cdot \partial y^2} = 2A_{mn} \cdot \frac{m^2 \cdot n^2 \cdot \pi^4}{a^2 \cdot b^2} \cdot \sin\frac{m \cdot \pi \cdot x}{a} \cdot \sin\frac{n \cdot \pi \cdot y}{b}
$$

$$
\frac{\partial^4 w_{mn}}{\partial y^4} = A_{mn} \cdot \frac{m^4 \cdot \pi^4}{b^4} \cdot \sin\frac{m \cdot \pi \cdot x}{a} \cdot \sin\frac{n \cdot \pi \cdot y}{b}
$$

$$\frac{\partial^4 w_{mn}}{\partial x^4} + 2 \cdot \frac{\partial^4 w_{mn}}{\partial x^2 \cdot \partial y^2} + \frac{\partial^4 w_{mn}}{\partial y^4} =$$

$$= A_{mn} \cdot \pi^4 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 \cdot \sin \frac{m \cdot \pi \cdot x}{a} \cdot \sin \frac{n \cdot \pi \cdot y}{b}$$

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \cdot \pi^4 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 \cdot \sin \frac{m \cdot \pi \cdot x}{a} \cdot \sin \frac{n \cdot \pi \cdot y}{b} = \frac{1}{k} \cdot Z(x,y)$$

$$Z(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Z_{mn} \cdot \sin \frac{m \cdot \pi \cdot x}{a} \cdot \sin \frac{n \cdot \pi \cdot y}{b}$$

$$\int_0^a \int_0^b Z(x,y)\cdot \sin\frac{r\cdot\pi\cdot x}{a}\cdot \sin\frac{s\cdot\pi\cdot y}{b}\cdot dx\cdot dy =$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} Z_{mn}\cdot \int_0^a \int_0^b \sin\frac{m\cdot\pi\cdot x}{a}\cdot \sin\frac{r\cdot\pi\cdot x}{a}\cdot \sin\frac{n\cdot\pi\cdot y}{b}\cdot \sin\frac{s\cdot\pi\cdot y}{b}\cdot dx\cdot dy$$

$$\int_0^a \sin\frac{m\cdot\pi\cdot x}{a}\cdot \sin\frac{r\cdot\pi\cdot x}{a}\cdot dx\cdot dy = \begin{cases} \dfrac{a}{2} & \text{za } m = r \\ 0 & \text{za } m \neq r \end{cases}$$

$$\int_0^b \sin\frac{n\cdot\pi\cdot y}{b}\cdot \sin\frac{s\cdot\pi\cdot y}{b}\cdot dx\cdot dy = \begin{cases} \dfrac{b}{2} & \text{za } n = s \\ 0 & \text{za } n \neq s \end{cases}$$

$$\int_0^a \int_0^b Z(x,y)\cdot \sin\frac{m\cdot\pi\cdot x}{a}\cdot \sin\frac{n\cdot\pi\cdot y}{b}\cdot dx\cdot dy = \frac{a\cdot b}{4}\cdot Z_{mn}$$

$$Z_{mn} = \frac{4}{a\cdot b}\cdot \int_0^a \int_0^b Z(x,y)\cdot \sin\frac{m\cdot\pi\cdot x}{a}\cdot \sin\frac{n\cdot\pi\cdot y}{b}\cdot dx\cdot dy$$

$$A_{mn} = \frac{Z_{mn}}{k\cdot\pi^4\cdot\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2}$$

$$A_{mn} = \frac{4}{a\cdot b}\cdot \frac{\int_0^a \int_0^b Z(x,y)\cdot \sin\frac{m\cdot\pi\cdot x}{a}\cdot \sin\frac{n\cdot\pi\cdot y}{b}\cdot dx\cdot dy}{k\cdot\pi^4\cdot\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2}$$

$$w(x,y) = \frac{4}{k \cdot a \cdot b \cdot \pi^4} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\int_0^a \int_0^b Z(x,y) \cdot \sin \frac{m \cdot \pi \cdot x}{a} \cdot \sin \frac{n \cdot \pi \cdot y}{b} \cdot dx \cdot dy}{\left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2} \cdot$$

$$\cdot \sin \frac{m \cdot \pi \cdot x}{a} \cdot \sin \frac{n \cdot \pi \cdot y}{b}$$

Jednakopodeljeno opterećenje

$$Z_0 \cdot \int_0^a \int_0^b \sin\frac{m \cdot \pi \cdot x}{a} \cdot \sin\frac{n \cdot \pi \cdot y}{b} \cdot dx \cdot dy =$$

$$= \frac{a \cdot b}{\pi^2} \cdot \frac{Z_0}{m \cdot n} \cdot (1 - \cos m\pi) \cdot (1 - \cos n\pi)$$

- parno "m" i "n"

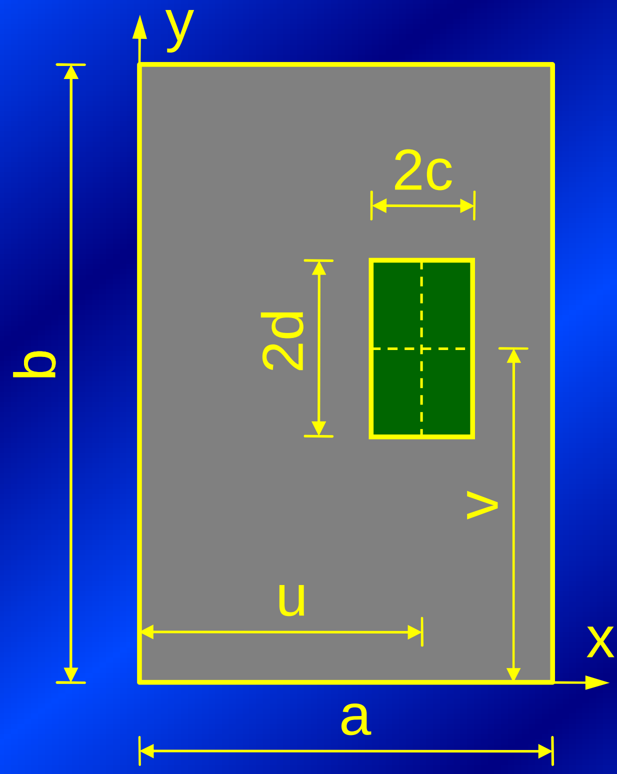
$$A_{mn} = 0$$

- neparno "m" i "n"

$$A_{mn} = \frac{16 \cdot Z_0}{k \cdot \pi^6 \cdot \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2}$$

$$w(x, y) = \frac{16 \cdot Z_0}{k \cdot \pi^6} \cdot \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m \cdot n \cdot \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2} \cdot \sin\frac{m \cdot \pi \cdot x}{a} \cdot \sin\frac{n \cdot \pi \cdot y}{b}$$

Jednakopodeljeno opterećenje na pravougaoniku



$$Z(x,y) = Z_0 \qquad \begin{cases} u-c < x < u+c \\ v-d < y < v+d \end{cases}$$

$$Z_{mn} = \frac{4 \cdot Z_0}{a \cdot b} \int_{u-c}^{u+c} \int_{v-d}^{v+d} \sin \frac{m \cdot \pi \cdot x}{a} \cdot$$

$$\cdot \sin \frac{n \cdot \pi \cdot y}{b} \cdot dx \cdot dy$$

$$Z(x,y)=\frac{16\cdot Z_0}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{1}{m\cdot n\cdot}\cdot\sin\frac{m\cdot\pi\cdot u}{a}\cdot\sin\frac{m\cdot\pi\cdot c}{a}\cdot\right.$$

$$\left.\cdot\sin\frac{n\cdot\pi\cdot v}{b}\cdot\sin\frac{n\cdot\pi\cdot d}{b}\right]\cdot\sin\frac{m\cdot\pi\cdot x}{a}\cdot\sin\frac{n\cdot\pi\cdot y}{b}$$

$$w(x,y)=\frac{16\cdot Z_0}{k\cdot\pi^6}\cdot\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\frac{m\cdot\pi\cdot u}{a}\cdot\sin\frac{m\cdot\pi\cdot c}{a}\cdot\sin\frac{n\cdot\pi\cdot v}{b}\cdot\sin\frac{n\cdot\pi\cdot d}{b}}{m\cdot n\cdot\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2}\cdot$$

$$\cdot\sin\frac{m\cdot\pi\cdot x}{a}\cdot\sin\frac{n\cdot\pi\cdot y}{b}$$

Koncentrisana sila P = $Z_0 \cdot 4 \cdot c \cdot d$

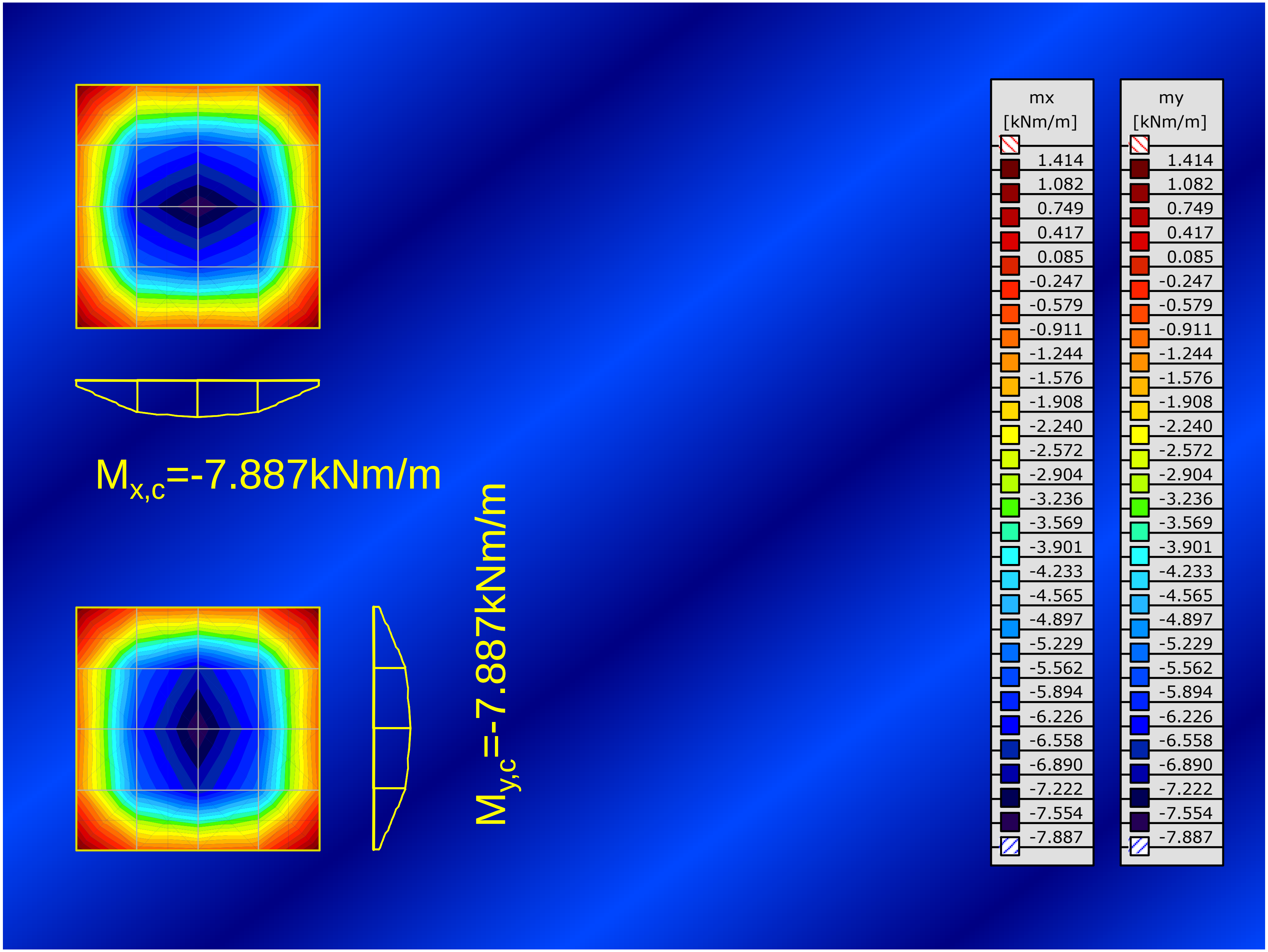
$$w(x,y) = \frac{4}{k \cdot a \cdot b \cdot \pi^4} \cdot 4 \cdot c \cdot d \cdot Z_0 \cdot \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin\frac{m \cdot \pi \cdot u}{a} \cdot \sin\frac{n \cdot \pi \cdot v}{b}}{\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2} \cdot$$

$$\cdot \frac{\sin\frac{m \cdot \pi \cdot c}{a}}{\frac{m \cdot \pi \cdot c}{a}} \cdot \frac{\sin\frac{n \cdot \pi \cdot d}{b}}{\frac{n \cdot \pi \cdot d}{b}} \cdot \sin\frac{m \cdot \pi \cdot x}{a} \cdot \sin\frac{n \cdot \pi \cdot y}{b}$$

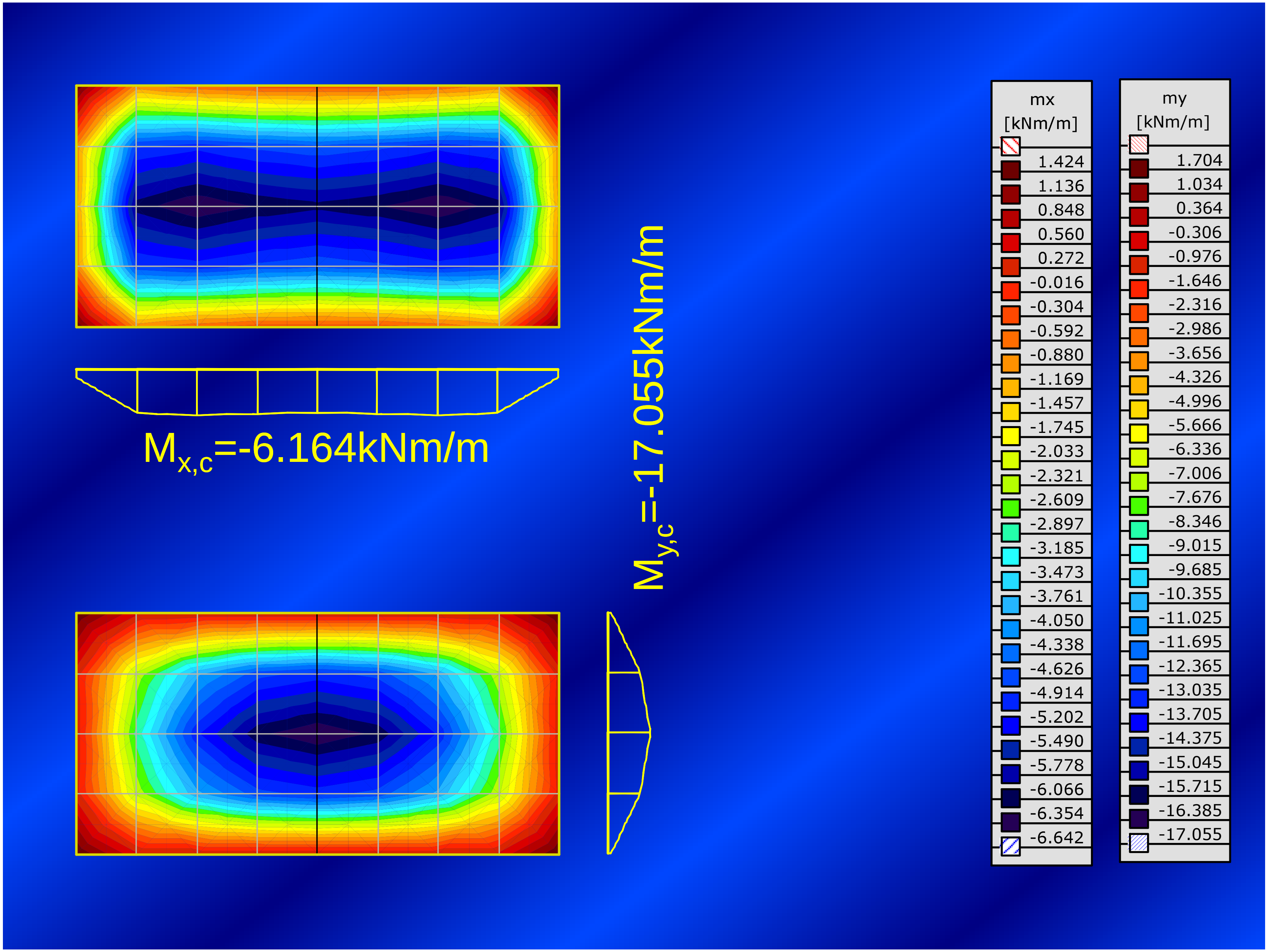
$$\lim_{c\to 0}\frac{\sin\dfrac{m\cdot\pi\cdot c}{a}}{\dfrac{m\cdot\pi\cdot c}{a}}=1 \qquad \lim_{d\to 0}\frac{\sin\dfrac{n\cdot\pi\cdot d}{b}}{\dfrac{n\cdot\pi\cdot d}{b}}=1$$

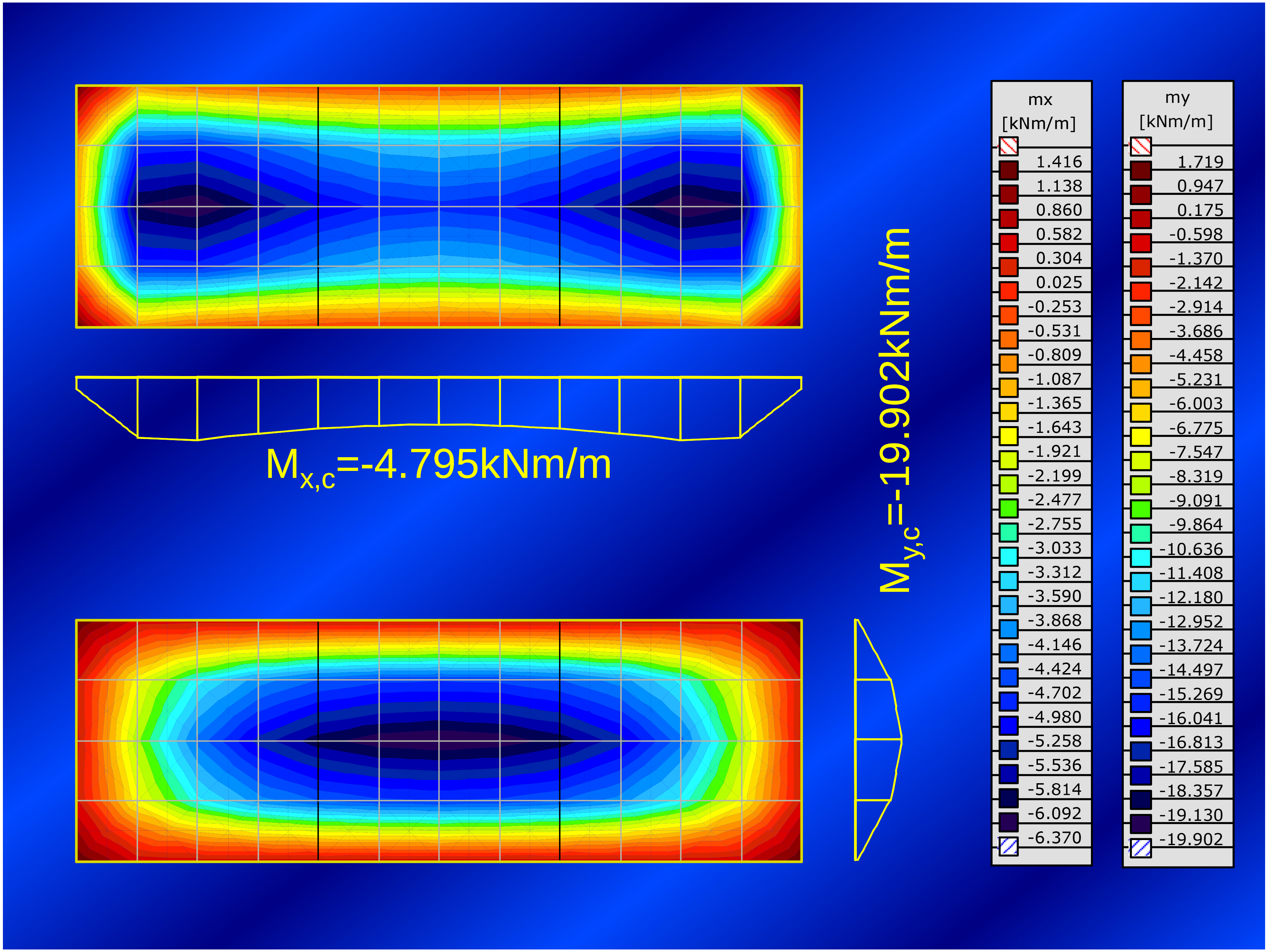
$$w(x,y)=\frac{4\cdot P}{k\cdot a\cdot b\cdot\pi^4}\cdot\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\dfrac{m\cdot\pi\cdot u}{a}\cdot\sin\dfrac{n\cdot\pi\cdot v}{b}}{\left(\dfrac{m^2}{a^2}+\dfrac{n^2}{b^2}\right)^2}\cdot$$

$$\cdot\sin\frac{m\cdot\pi\cdot x}{a}\cdot\sin\frac{n\cdot\pi\cdot y}{b}$$

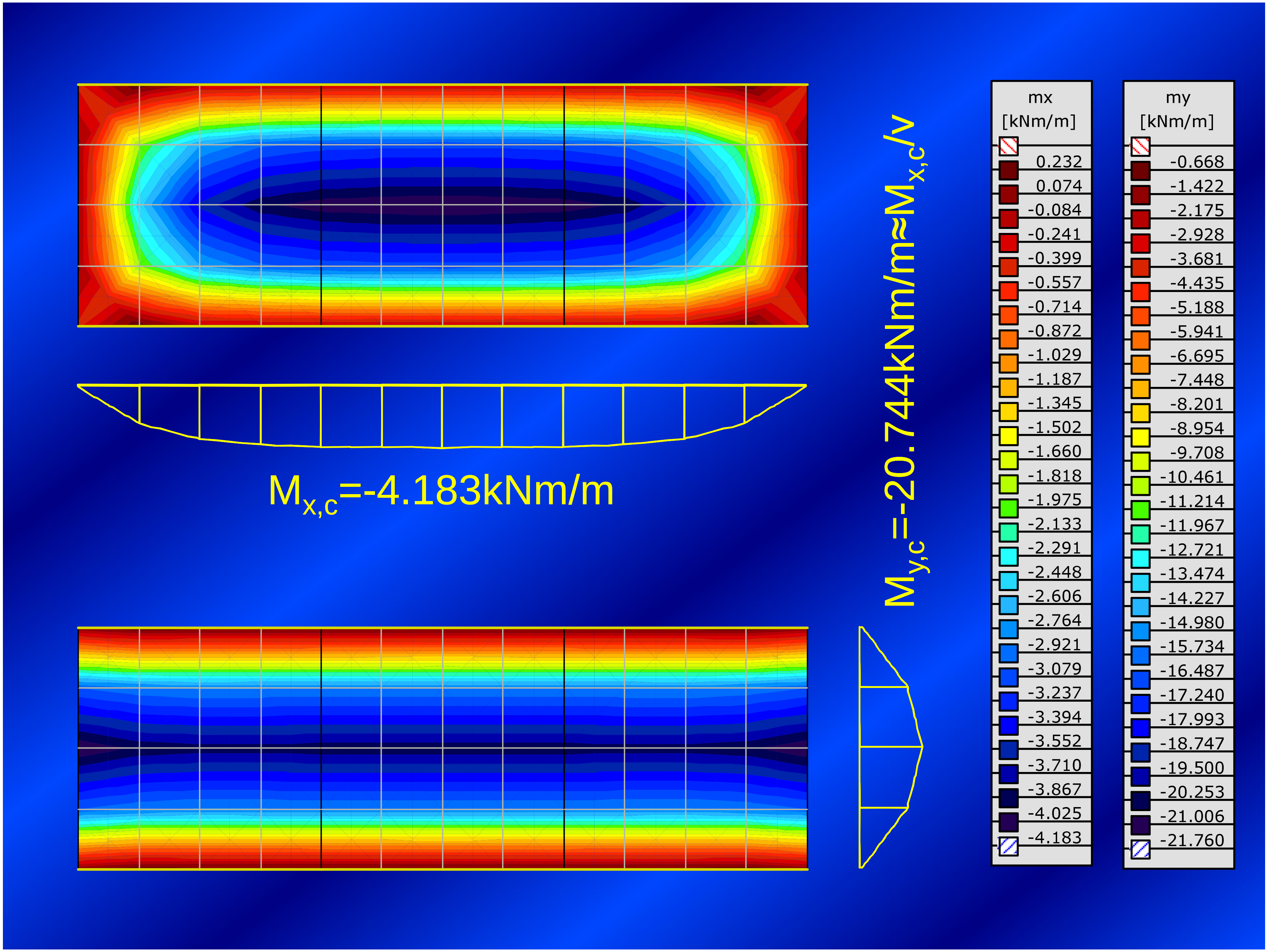


$M_{x,c}$=-7.887kNm/m
$M_{y,c}$=-7.887kNm/m
mx
[kNm/m]
1.414
1.082
0.749
0.417
0.085
-0.247
-0.579
-0.911
-1.244
-1.576
-1.908
-2.240
-2.572
-2.904
-3.236
-3.569
-3.901
-4.233
-4.565
-4.897
-5.229
-5.562
-5.894
-6.226
-6.558
-6.890
-7.222
-7.554
-7.887
my
[kNm/m]
1.414
1.082
0.749
0.417
0.085
-0.247
-0.579
-0.911
-1.244
-1.576
-1.908
-2.240
-2.572
-2.904
-3.236
-3.569
-3.901
-4.233
-4.565
-4.897
-5.229
-5.562
-5.894
-6.226
-6.558
-6.890
-7.222
-7.554
-7.887


$M_{x,c}$=-6.164kNm/m
$M_{y,c}$=-17.055kNm/m
mx [kNm/m]
1.424
1.136
0.848
0.560
0.272
-0.016
-0.304
-0.592
-0.880
-1.169
-1.457
-1.745
-2.033
-2.321
-2.609
-2.897
-3.185
-3.473
-3.761
-4.050
-4.338
-4.626
-4.914
-5.202
-5.490
-5.778
-6.066
-6.354
-6.642
my [kNm/m]
1.704
1.034
0.364
-0.306
-0.976
-1.646
-2.316
-2.986
-3.656
-4.326
-4.996
-5.666
-6.336
-7.006
-7.676
-8.346
-9.015
-9.685
-10.355
-11.025
-11.695
-12.365
-13.035
-13.705
-14.375
-15.045
-15.715
-16.385
-17.055

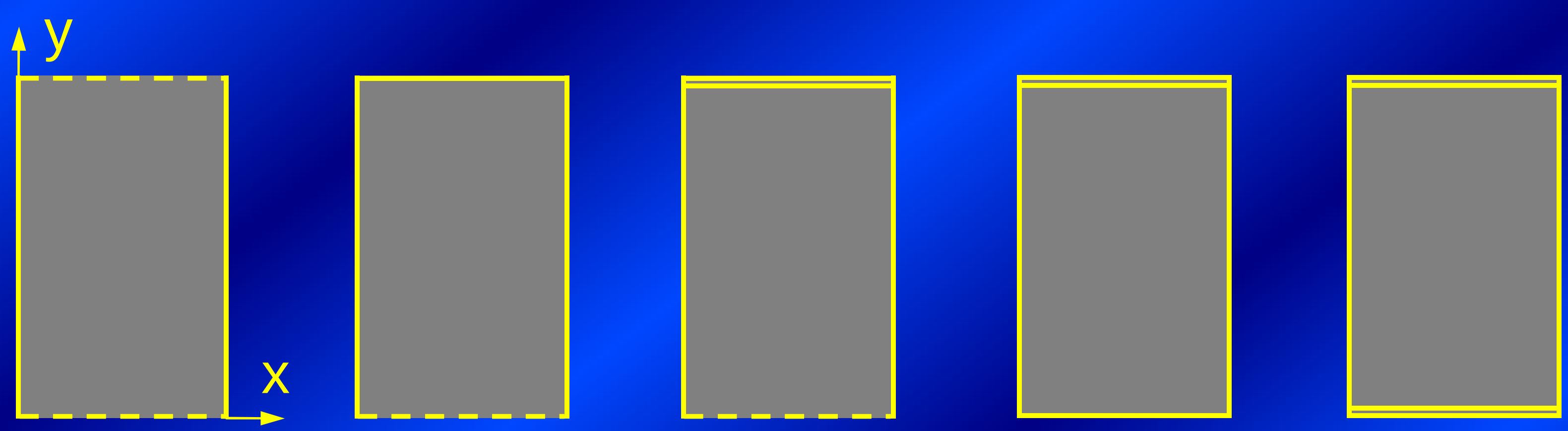


$M_{x,c}$=-4.795kNm/m
$M_{y,c}$=-19.902kNm/m
mx [kNm/m]
1.416
1.138
0.860
0.582
0.304
0.025
-0.253
-0.531
-0.809
-1.087
-1.365
-1.643
-1.921
-2.199
-2.477
-2.755
-3.033
-3.312
-3.590
-3.868
-4.146
-4.424
-4.702
-4.980
-5.258
-5.536
-5.814
-6.092
-6.370
my [kNm/m]
1.719
0.947
0.175
-0.598
-1.370
-2.142
-2.914
-3.686
-4.458
-5.231
-6.003
-6.775
-7.547
-8.319
-9.091
-9.864
-10.636
-11.408
-12.180
-12.952
-13.724
-14.497
-15.269
-16.041
-16.813
-17.585
-18.357
-19.130
-19.902



mx
[kNm/m]
0.232
0.074
-0.084
-0.241
-0.399
-0.557
-0.714
-0.872
-1.029
-1.187
-1.345
-1.502
-1.660
-1.818
-1.975
-2.133
-2.291
-2.448
-2.606
-2.764
-2.921
-3.079
-3.237
-3.394
-3.552
-3.710
-3.867
-4.025
-4.183
my
[kNm/m]
-0.668
-1.422
-2.175
-2.928
-3.681
-4.435
-5.188
-5.941
-6.695
-7.448
-8.201
-8.954
-9.708
-10.461
-11.214
-11.967
-12.721
-13.474
-14.227
-14.980
-15.734
-16.487
-17.240
-17.993
-18.747
-19.500
-20.253
-21.006
-21.760
$M_{x,c}$=-4.183kNm/m
$M_{y,c}$=-20.744kNm/m≈$M_{x,c}$/v

Pravougaone ploče sa paralelnim slobodno oslonjenim ivicama (Maurice Lévy, 1899)

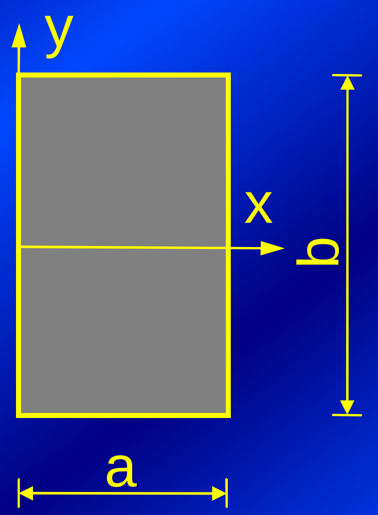


$$w = w_h + w_p \qquad w_h = \sum_{n=1}^{\infty} Y_n \cdot \sin\frac{n\cdot\pi\cdot x}{a} \quad \rightarrow \quad \Delta\Delta w_h = 0$$

$$Y_n^{IV} - 2\cdot\frac{n^2\cdot\pi^2}{a^2}\cdot Y_n^{II} + \frac{n^4\cdot\pi^4}{a^4}\cdot Y_n = 0$$

$$Y_n = \left(A_n + \frac{n\cdot\pi\cdot y}{a}\cdot B_n\right)\cdot \mathrm{ch}\frac{n\cdot\pi\cdot y}{a} + \left(C_n + \frac{n\cdot\pi\cdot y}{a}\cdot D_n\right)\cdot \mathrm{sh}\frac{n\cdot\pi\cdot y}{a}$$

Jednakopodeljeno opterećenje



$$w_p = \frac{Z_0}{24 \cdot k} \cdot (x^4 - 2a \cdot x^3 + a^3 \cdot x)$$

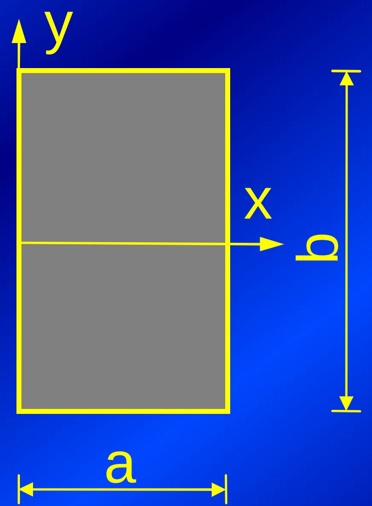
$$w_p = 4 \cdot \frac{Z_0 \cdot a^4}{\pi^5 \cdot k} \sum_{n=1}^{\infty} \frac{1}{n^5} \cdot \sin \frac{n \cdot \pi \cdot x}{a}$$

$$w_{h,sim} = \sum_{n=1}^{\infty} \left(A_n \cdot \mathrm{ch} \frac{n \cdot \pi \cdot y}{a} + D_n \cdot \frac{n \cdot \pi \cdot y}{a} \cdot \mathrm{sh} \frac{n \cdot \pi \cdot y}{a} \right) \cdot \sin \frac{n \cdot \pi \cdot x}{a}$$

$$w_{h,ant} = \sum_{n=1}^{\infty} \left(B_n \cdot \frac{n \cdot \pi \cdot y}{a} \cdot \mathrm{ch} \frac{n \cdot \pi \cdot y}{a} + C_n \cdot \mathrm{sh} \frac{n \cdot \pi \cdot y}{a} \right) \cdot \sin \frac{n \cdot \pi \cdot x}{a}$$

Ploča slobodno oslonjena na ivici y=±b/2

$$w = 0 \qquad \frac{\partial^2 w}{\partial y^2} = 0$$



$$w_n = \left(4 \cdot \frac{Z_0 \cdot a^4}{\pi^5 \cdot k \cdot n^5} + A_n \cdot \mathrm{ch}\frac{n \cdot \pi \cdot y}{a} + D_n \cdot \frac{n \cdot \pi \cdot y}{a} \cdot \mathrm{sh}\frac{n \cdot \pi \cdot y}{a}\right) \cdot \sin\frac{n \cdot \pi \cdot x}{a}$$

$$\frac{\partial^2 w_n}{\partial y^2} = \frac{n^2 \cdot \pi^2}{a^2} \cdot \left[A_n \cdot \mathrm{ch}\frac{n \cdot \pi \cdot y}{a} + D_n \cdot \left(2\mathrm{ch}\frac{n \cdot \pi \cdot y}{a} + \frac{n \cdot \pi \cdot y}{a} \cdot \mathrm{sh}\frac{n \cdot \pi \cdot y}{a}\right)\right] \cdot \sin\frac{n \cdot \pi \cdot x}{a}$$

$$A_n \cdot ch\alpha_n + \alpha_n \cdot D_n \cdot sh\alpha_n + \frac{S}{n^5} = 0 \qquad \alpha_n = \frac{n \cdot \pi \cdot b}{2a}$$

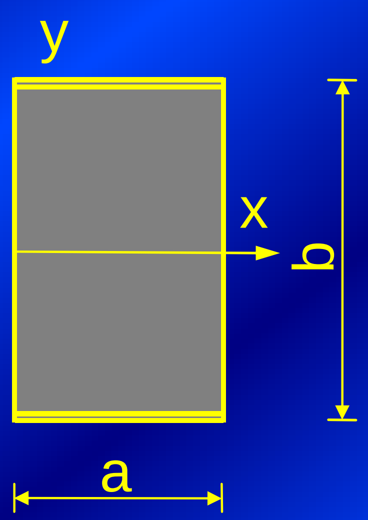
$$A_n \cdot ch\alpha_n + D_n \cdot (2 \cdot ch\alpha_n + \alpha_n \cdot sh\alpha_n) = 0 \qquad S = 4 \cdot \frac{Z_0 \cdot a^4}{\pi^5 \cdot k}$$

$$A_n = -\frac{1}{n^5} \cdot \frac{\alpha_n \cdot th\alpha_n + 2}{2 \cdot ch\alpha_n} \cdot S \qquad D_n = \frac{1}{n^5} \cdot \frac{S}{2 \cdot ch\alpha_n}$$

$$w = 4 \cdot \frac{Z_0 \cdot a^4}{\pi^5 \cdot k} \cdot \sum_{n=1}^{\infty} \frac{1}{n^5} \cdot \left(1 - \frac{\alpha_n \cdot th\alpha_n + 2}{2 \cdot ch\alpha_n} \cdot ch\frac{n \cdot \pi \cdot y}{a} + \right.$$

$$\left. + \frac{1}{2 \cdot ch\alpha_n} \cdot \frac{n \cdot \pi \cdot y}{a} \cdot sh\frac{n \cdot \pi \cdot y}{a} \right) \cdot \sin\frac{n \cdot \pi \cdot x}{a}$$

Ploča uklještena na ivici y=±b/2



$$A_n \cdot ch\alpha_n + \alpha_n \cdot D_n \cdot sh\alpha_n + \frac{S}{n^5} = 0$$

$$A_n \cdot sh\alpha_n + D_n \cdot (\alpha_n \cdot ch\alpha_n + sh\alpha_n) = 0$$

$$A_n = -\frac{1}{n^5} \cdot \frac{\alpha_n \cdot ch\alpha_n + sh\alpha_n}{\alpha_n + sh\alpha_n \cdot ch\alpha_n} \cdot S \qquad D_n = \frac{1}{n^5} \cdot \frac{sh\alpha_n}{\alpha_n + sh\alpha_n \cdot ch\alpha_n} \cdot S$$

$$w = 4 \cdot \frac{Z_0 \cdot a^4}{\pi^5 \cdot k} \cdot \sum_{n=1}^{\infty} \frac{1}{n^5} \cdot \left\{ 1 - \frac{sh\alpha_n}{\alpha_n + sh\alpha_n \cdot ch\alpha_n} \cdot \right.$$

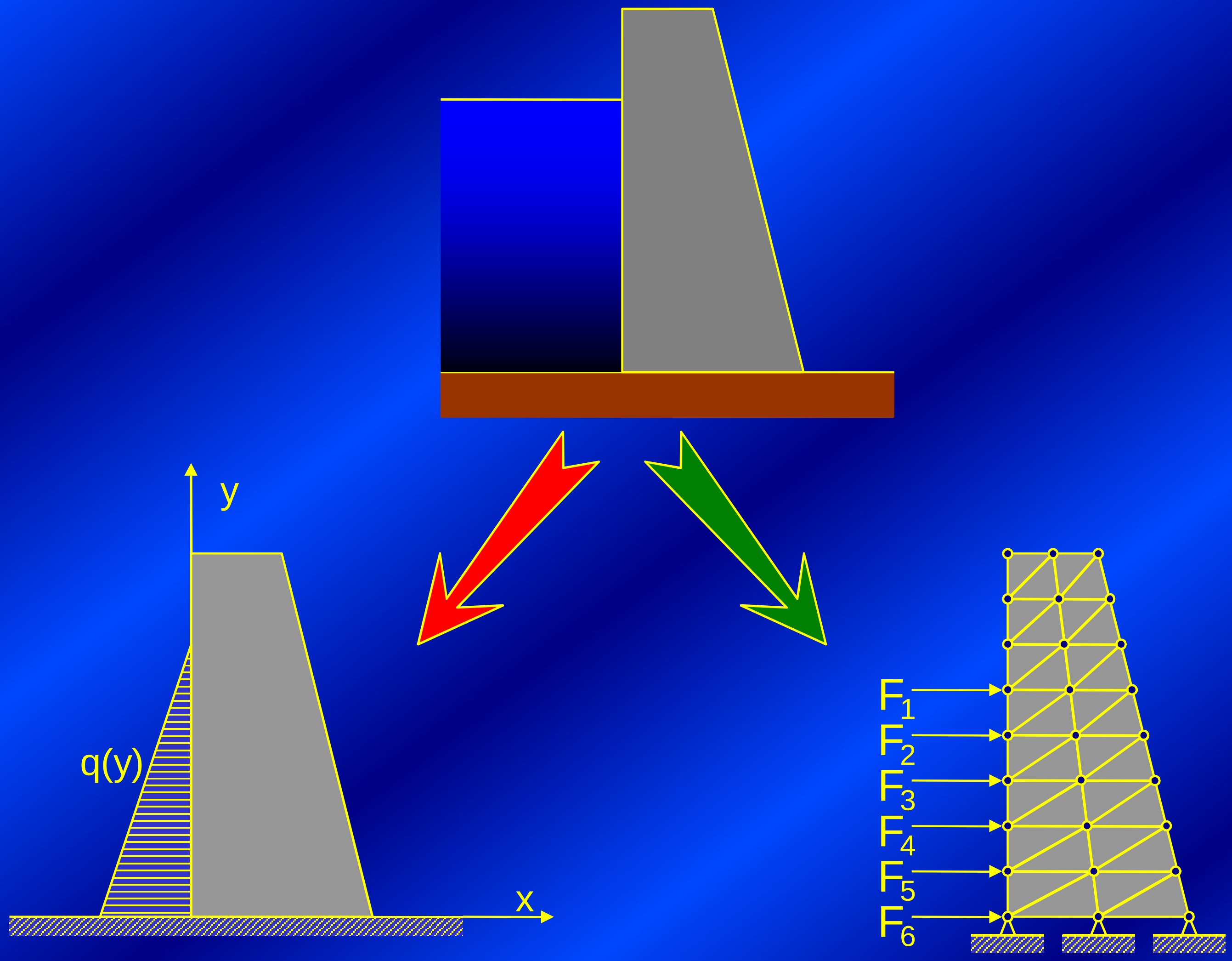
$$\left. \cdot \left[(\alpha_n \cdot ch\alpha_n + 1) \cdot ch\frac{n \cdot \pi \cdot y}{a} - \frac{n \cdot \pi \cdot y}{a} \cdot sh\frac{n \cdot \pi \cdot y}{a} \right] \right\} \cdot \sin\frac{n \cdot \pi \cdot x}{a}$$

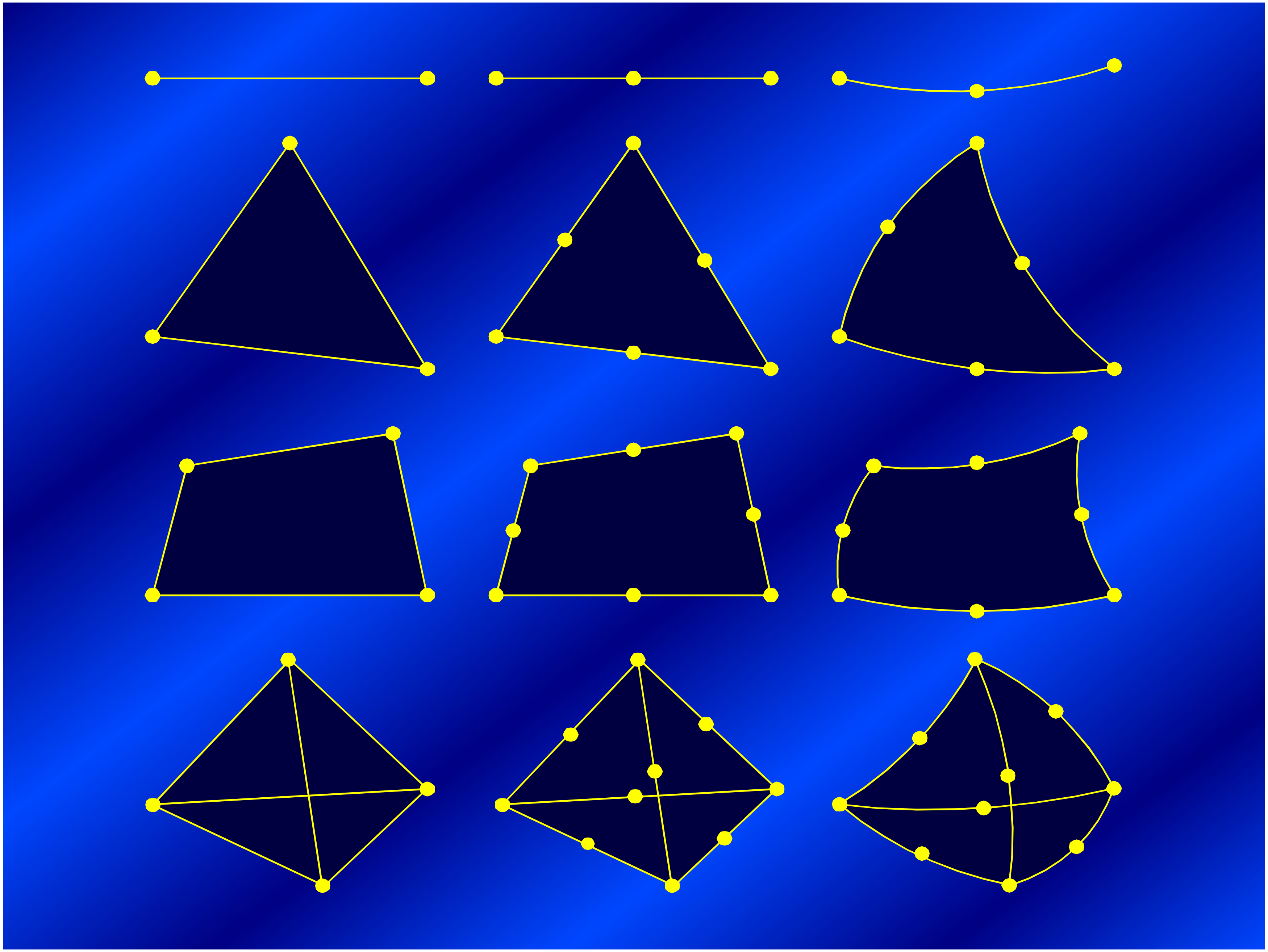
Numeričke metode za analizu i proračun ploča

- metode zasnovane na matematičkoj aproksimaciji (metoda konačnih razlika)
- metode zasnovane na fizičkoj diskretizaciji - modeliranju (metoda konačnih traka, metoda konačnih elemenata)

- Modeliranje konstrukcija - kreiranje idealizovane i pojednostavljene reprezentacije ponašanja konstrukcija - odlučujući korak u procesu projektovanja
- Greške i propusti u modeliranju mogu da budu uzrok ozbiljnih problema i teškoća u projektovanju
- Numeričko modeliranje je matematička realizacija izabranog koncepta modeliranja konstrukcije

Metoda konačnih elemenata (MKE)





Osobenosti MKE

- MKE je metoda diskretne analize - funkcija rešenja nije kontinualna - rešenje se dobija u čvorovima sistema KE - čvorovi su jedine veze između KE modela
- polje nepoznatih (pomeranja, na primer) u okviru pojedinačnog KE definisano je tzv. intepolacionim polinomom
- MKE je metoda "matrične" analize - - sistem MKE definisan je algebarskim umesto parcijalnim diferencijalnim jednačinama

Prednosti primene MKE

- dovoljna tačnost
- efikasno numeričko modeliranje
- jednostavna implementacija u CAA softveru

Jednačina ravnoteže KE:

$$[k] \cdot \{u\} = \{f\}$$

$$\{u\} = [k]^{-1} \cdot \{f\}$$

- [k] - matrica krutosti KE
- {u} - vektor pomeranja čvorova KE
- {f} - vektor sila čvorova KE

Matrica krutosti KE:

$$[k]=\int_V [B]^T\cdot[E]\cdot[B]\cdot dV$$

$$u(x,y,z)=[N]\cdot\{u\}$$

- [B] - matrica veza deformacija i pomeranja
- [E] - matrica karakteristika materijala
- [N] - matrica interpolacionih polinoma

Koordinatni sistemi

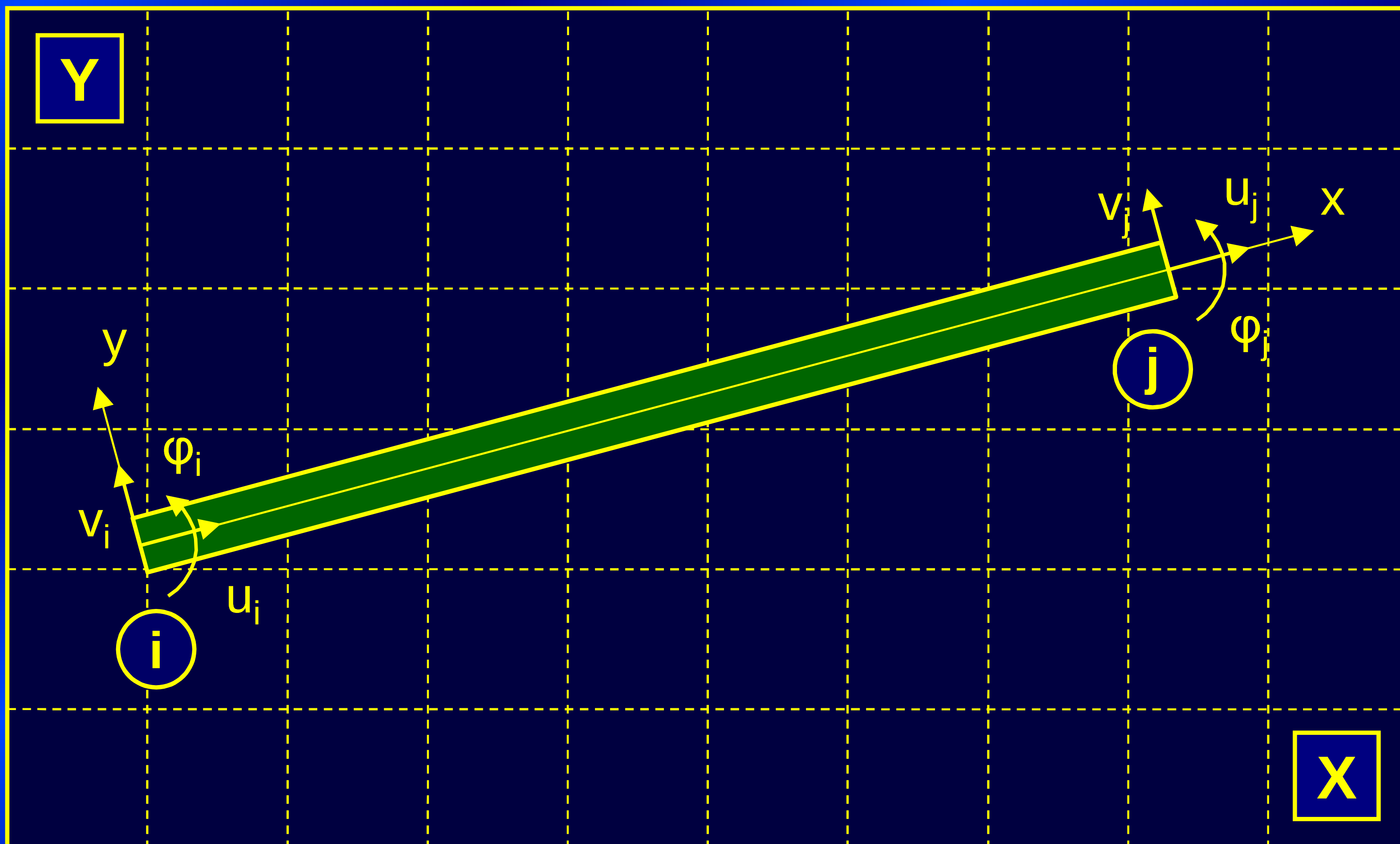
- Globalni koordinatni sistem sistema KE (koordinate čvorova, pomeranja čvorova, reakcije oslonaca i opterećenje sistema KE)
- Lokalni koordinatni sistem KE (sile u presecima, naponi dilatacije i opterećenje KE)

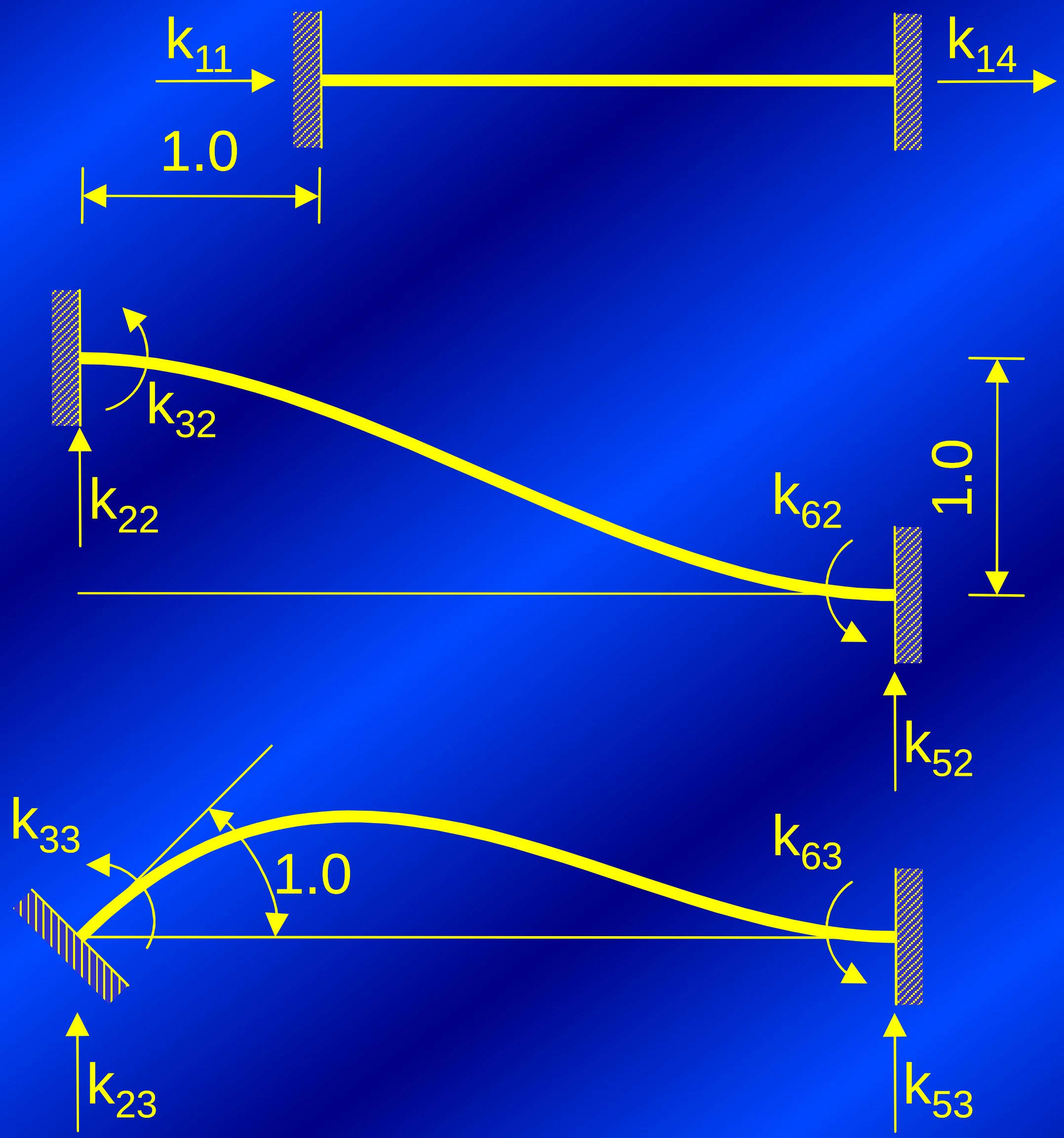
Transformacija lokalnih koordinata u globalne koordinate

$$[k_g]=[T]^T \cdot [k] \cdot [T]$$

- [T] - matrica transformacije koordinata
- [k] - matrica krutosti KE u lokalnom koordinatnom sistemu
- $[k_g]$ - matrica krutosti KE u globalnom koordinatnom sistemu

Globalni i lokalni koordinatni sistem štapa u ravni sa 6 stepeni slobode

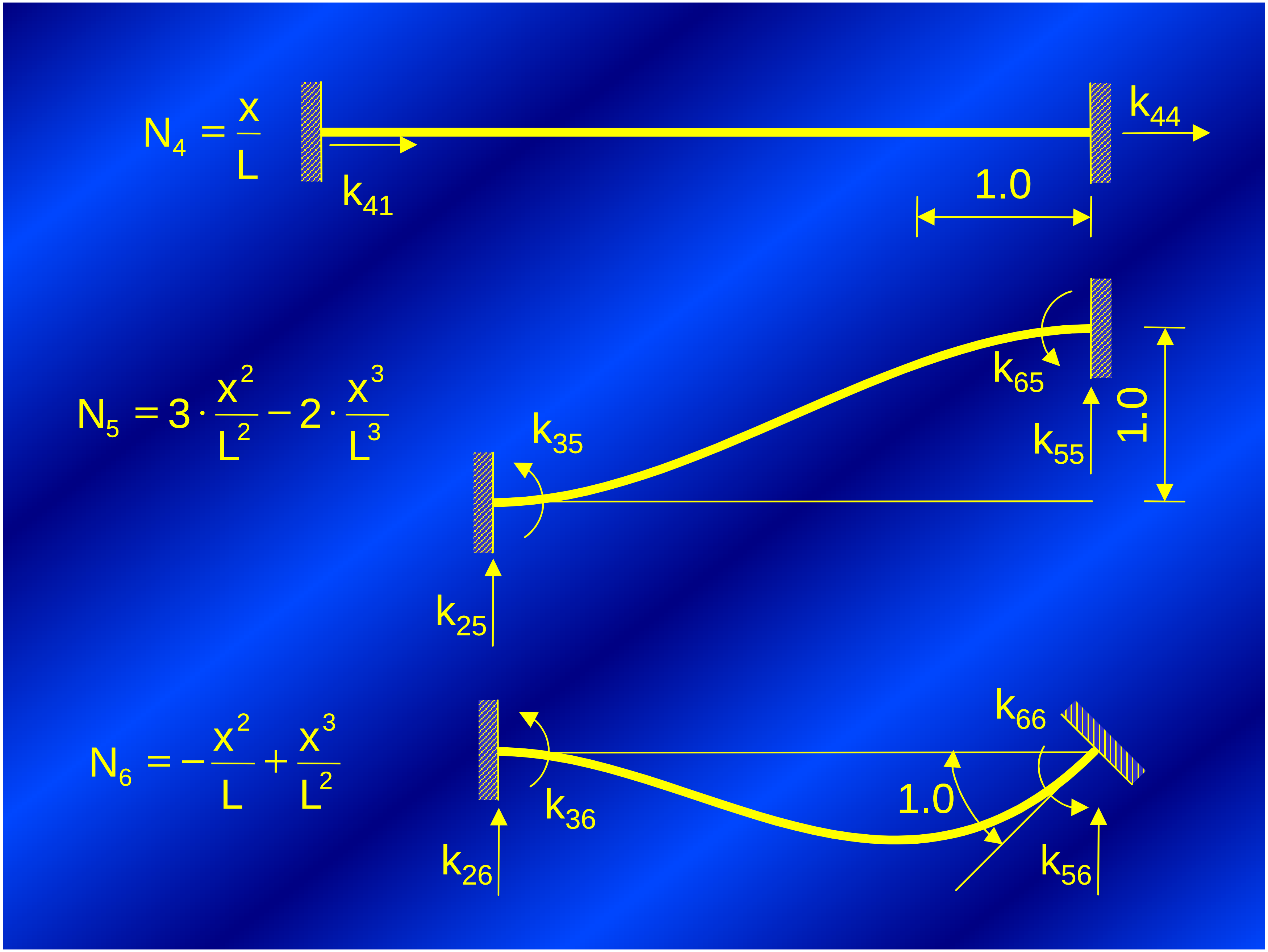




$$N_1 = 1 - \frac{x}{L}$$

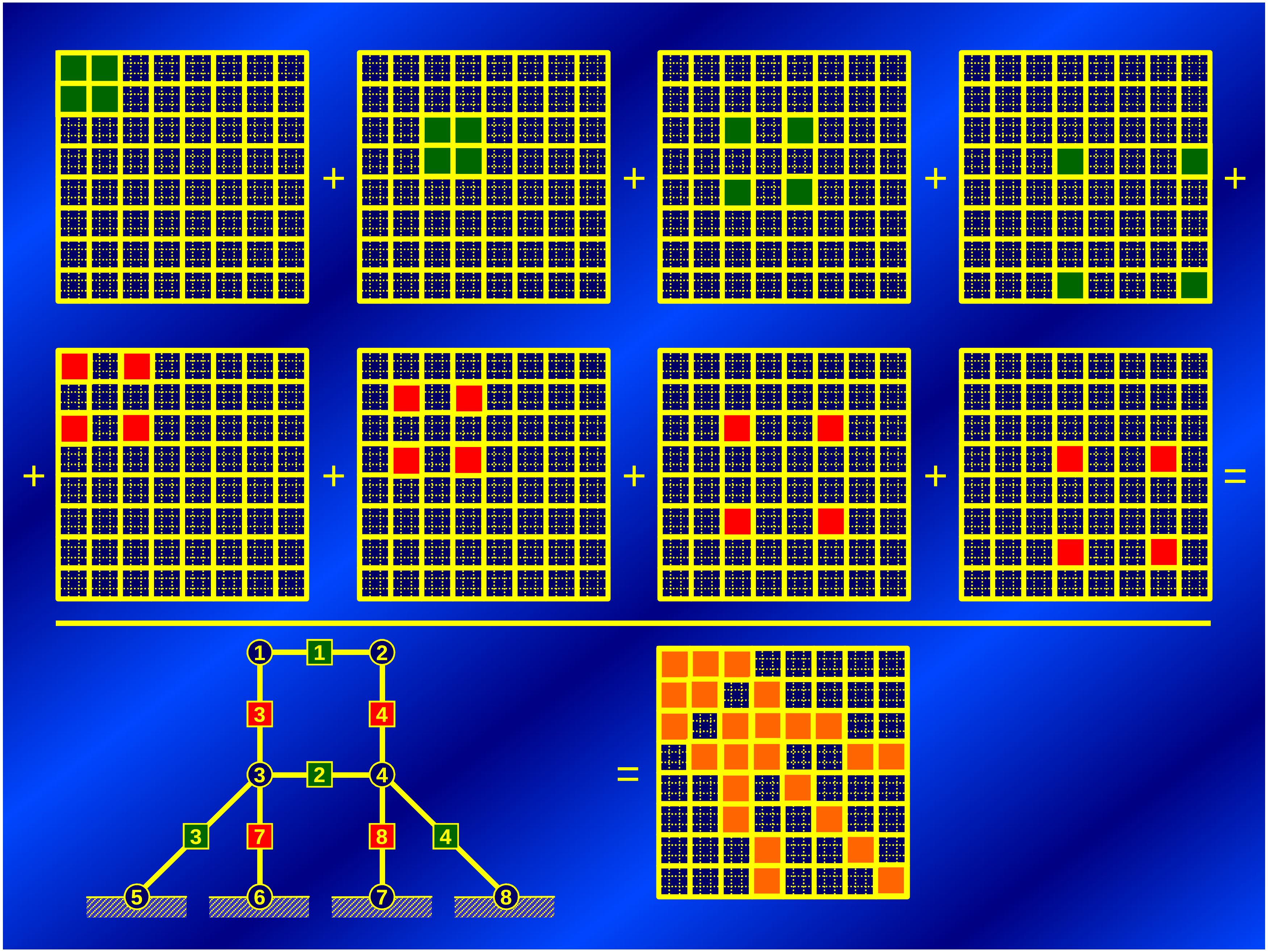
$$N_2 = 1 - 3 \cdot \frac{x^2}{L^2} + 2 \cdot \frac{x^3}{L^3}$$

$$N_3 = x - 2 \cdot \frac{x^2}{L} + \frac{x^3}{L^2}$$

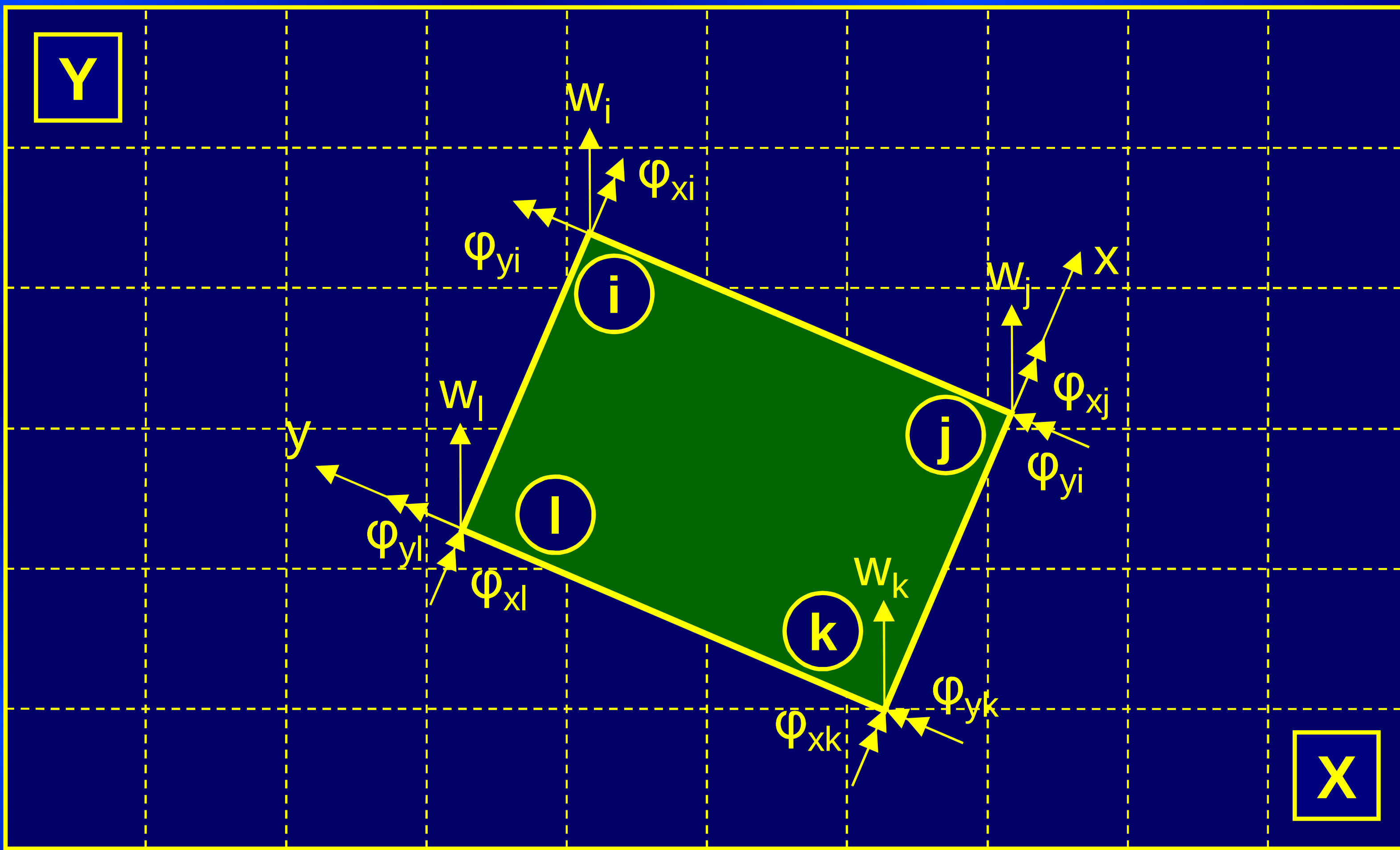

$N_4 = \frac{x}{L}$
k_{44}
k_{41}
1.0
$N_5 = 3 \cdot \frac{x^2}{L^2} - 2 \cdot \frac{x^3}{L^3}$
k_{65}
k_{35}
k_{55}
1.0
k_{25}
$N_6 = -\frac{x^2}{L} + \frac{x^3}{L^2}$
k_{66}
k_{36}
1.0
k_{26}
k_{56}

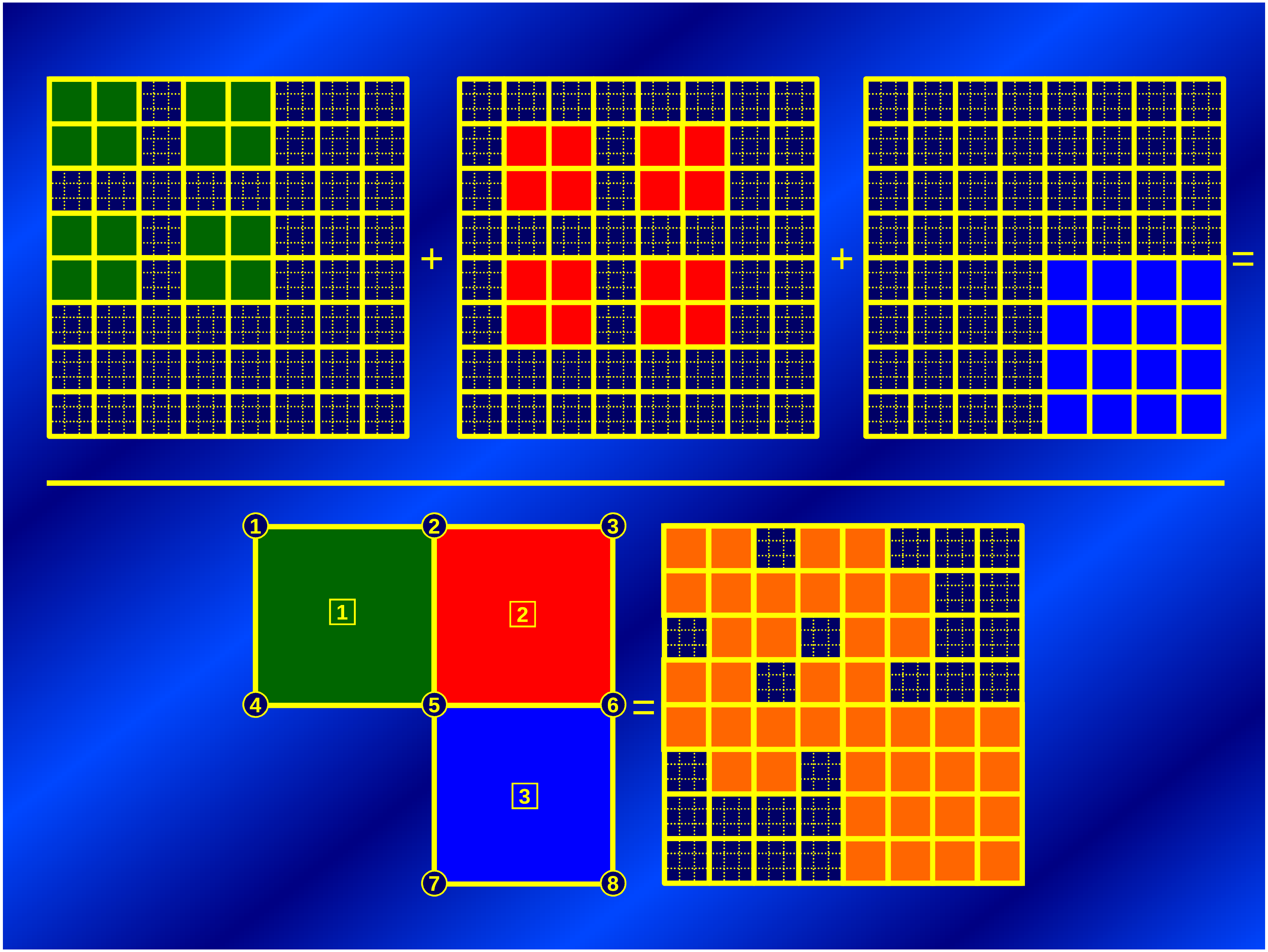
$$
[k] = \begin{bmatrix}
\frac{E \cdot A}{L} & 0 & 0 & -\frac{E \cdot A}{L} & 0 & 0 \\
 & \frac{12 \cdot E \cdot I}{L^3} & \frac{6 \cdot E \cdot I}{L^2} & 0 & -\frac{12 \cdot E \cdot I}{L^3} & -\frac{6 \cdot E \cdot I}{L^2} \\
 & & \frac{4 \cdot E \cdot I}{L} & 0 & -\frac{6E \cdot I}{L^2} & \frac{2 \cdot E \cdot I}{L} \\
 & & & \frac{E \cdot A}{L} & 0 & 0 \\
 & & & & \frac{12 \cdot E \cdot I}{L^3} & -\frac{6 \cdot E \cdot I}{L^2} \\
\text{sim.} & & & & & \frac{4 \cdot E \cdot I}{L}
\end{bmatrix}
$$

- E - modul elastičnosti materijala štapa [kN/m^2]
- A - površina poprečnog preseka štapa [m^2]
- L - raspon štapa [m]
- I - moment inercije poprečnog preseka štapa [m^4]

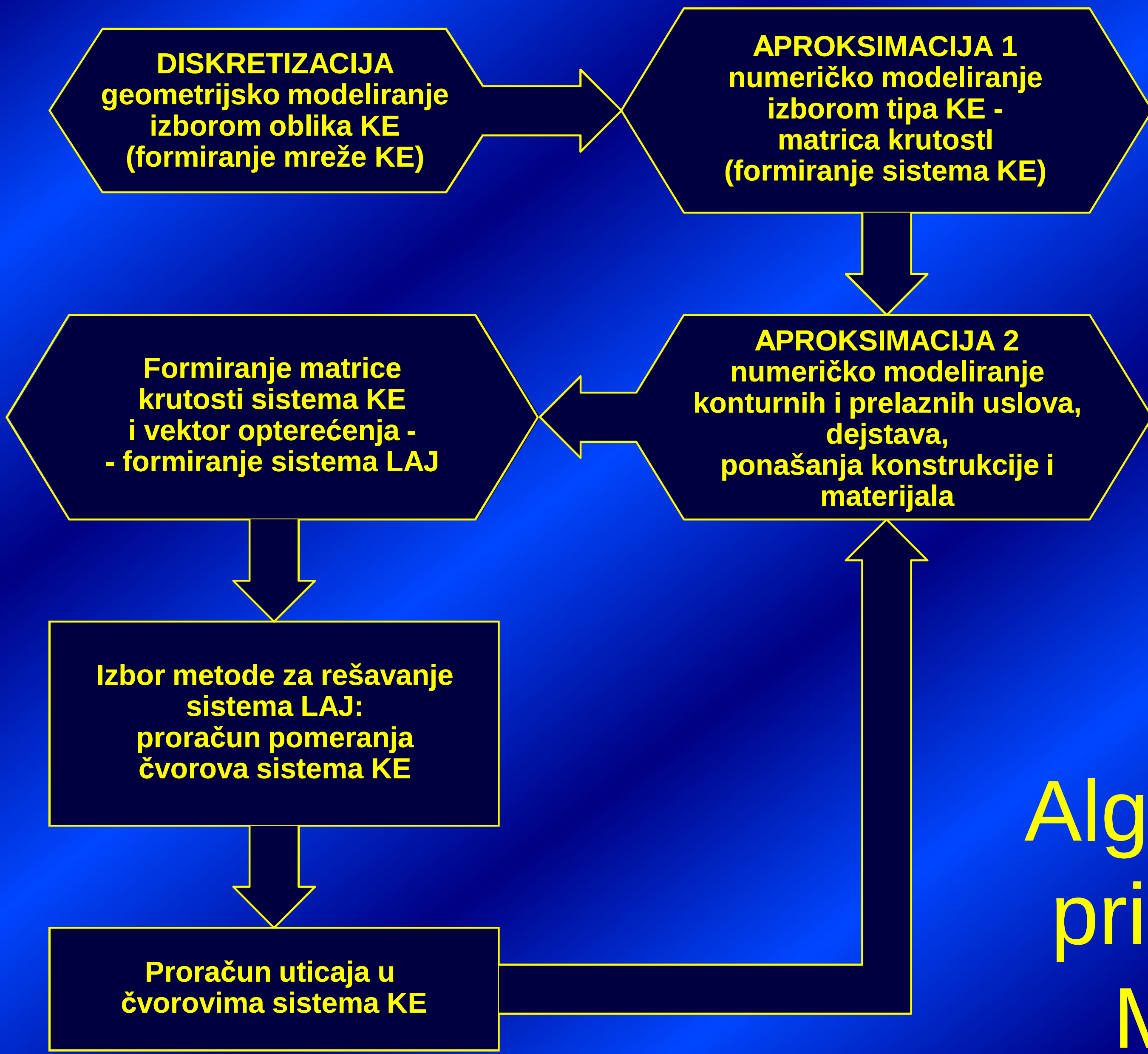

+
=
1
2
3
4
5
6
7
8
1
2
3
4
3
4
7
8

Globalni i lokalni koordinatni sistem četvorougaonog KE sa 12 stepeni slobode



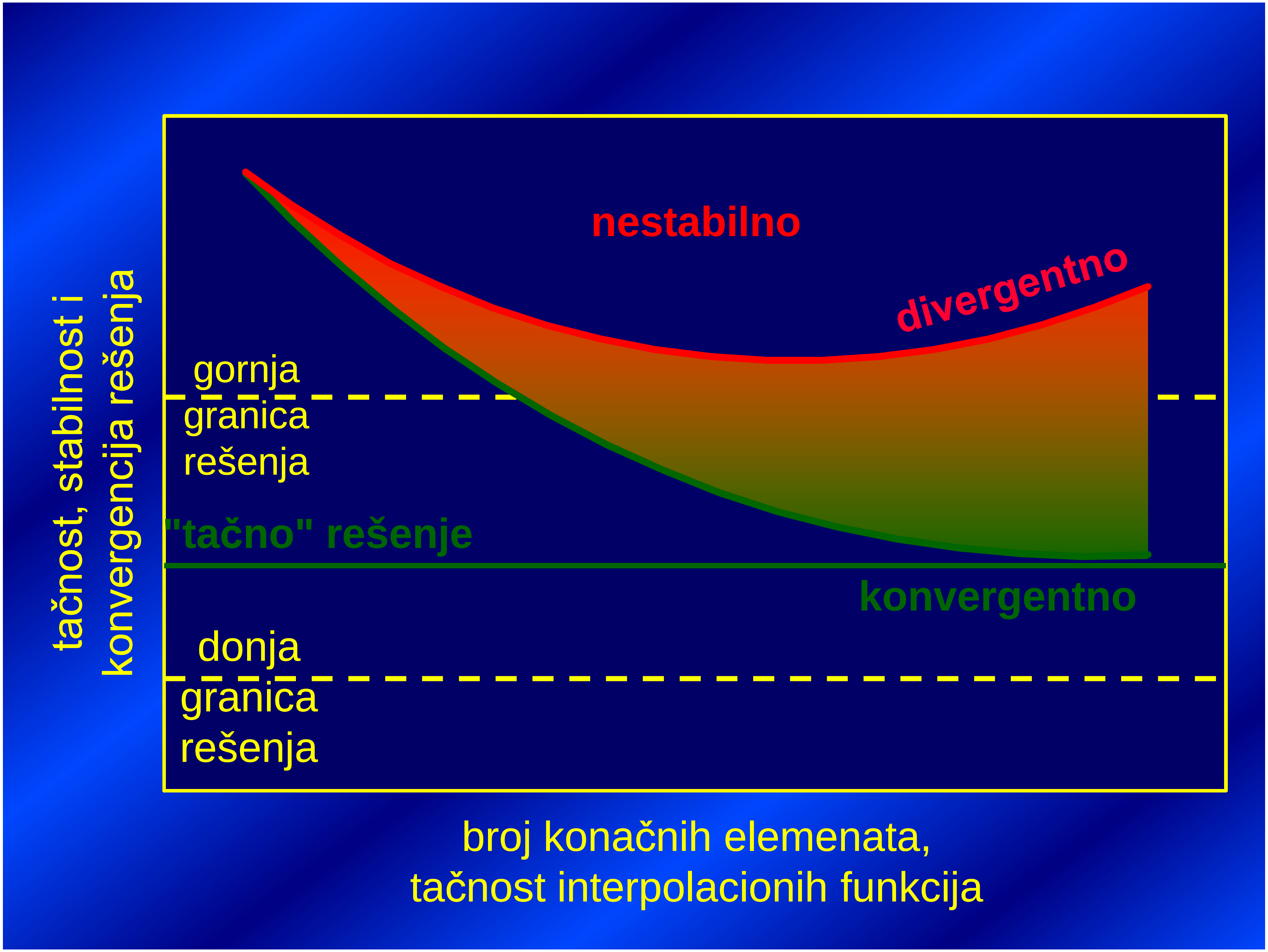

+
+
=
1
2
3
4
5
6
7
8
1
2
3
=

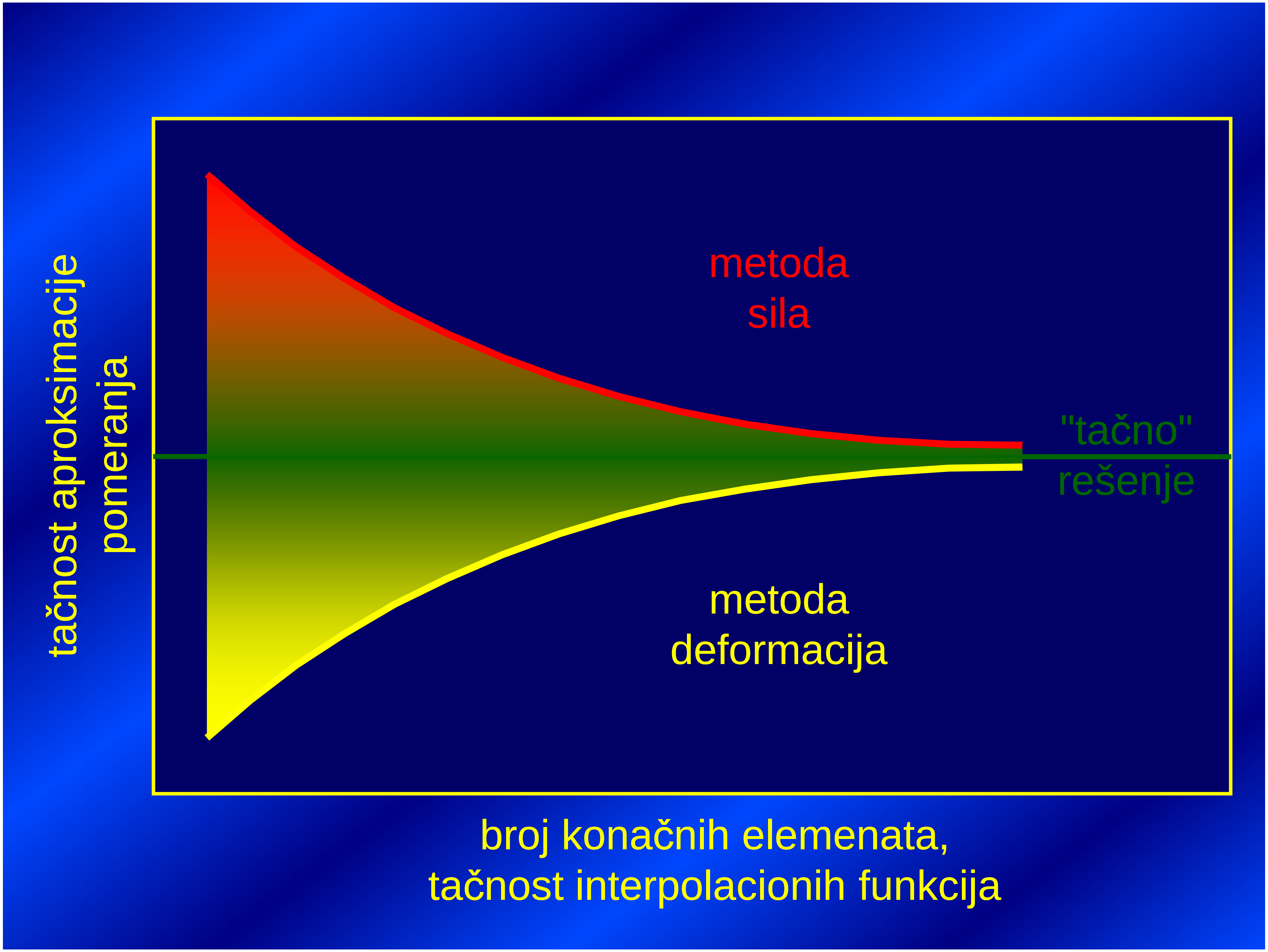
Algoritam primene MKE

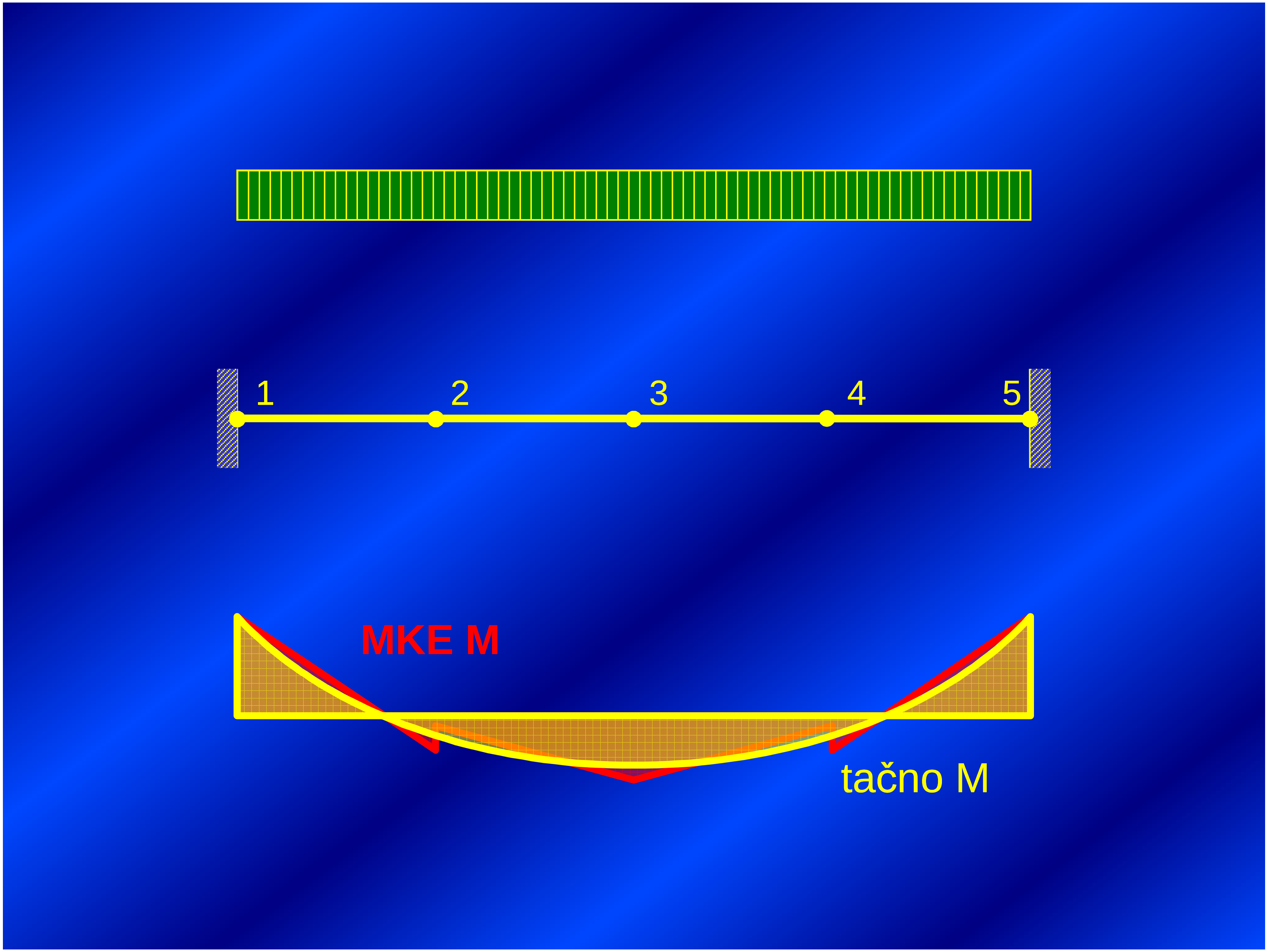


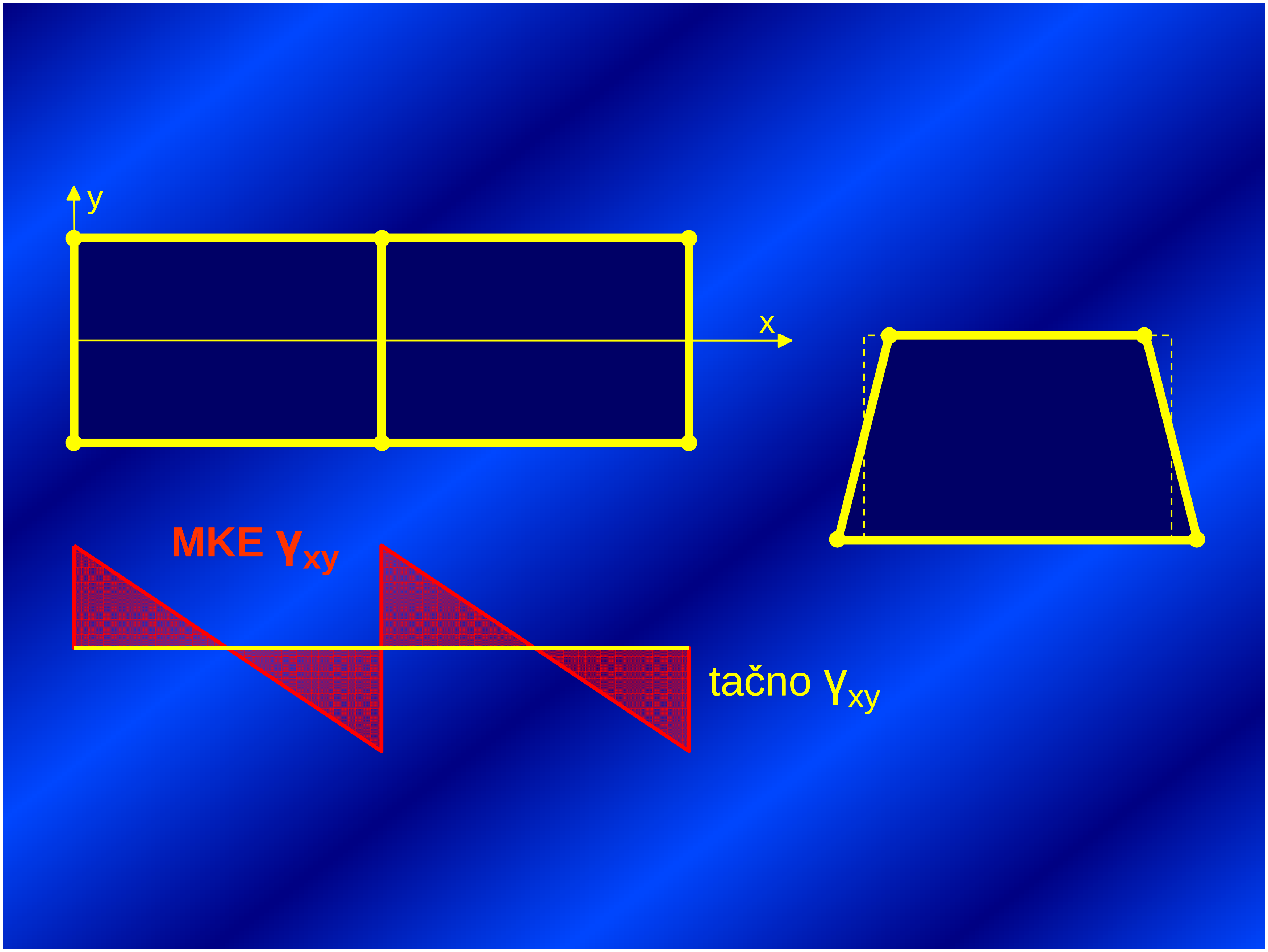
Greške u primeni MKE

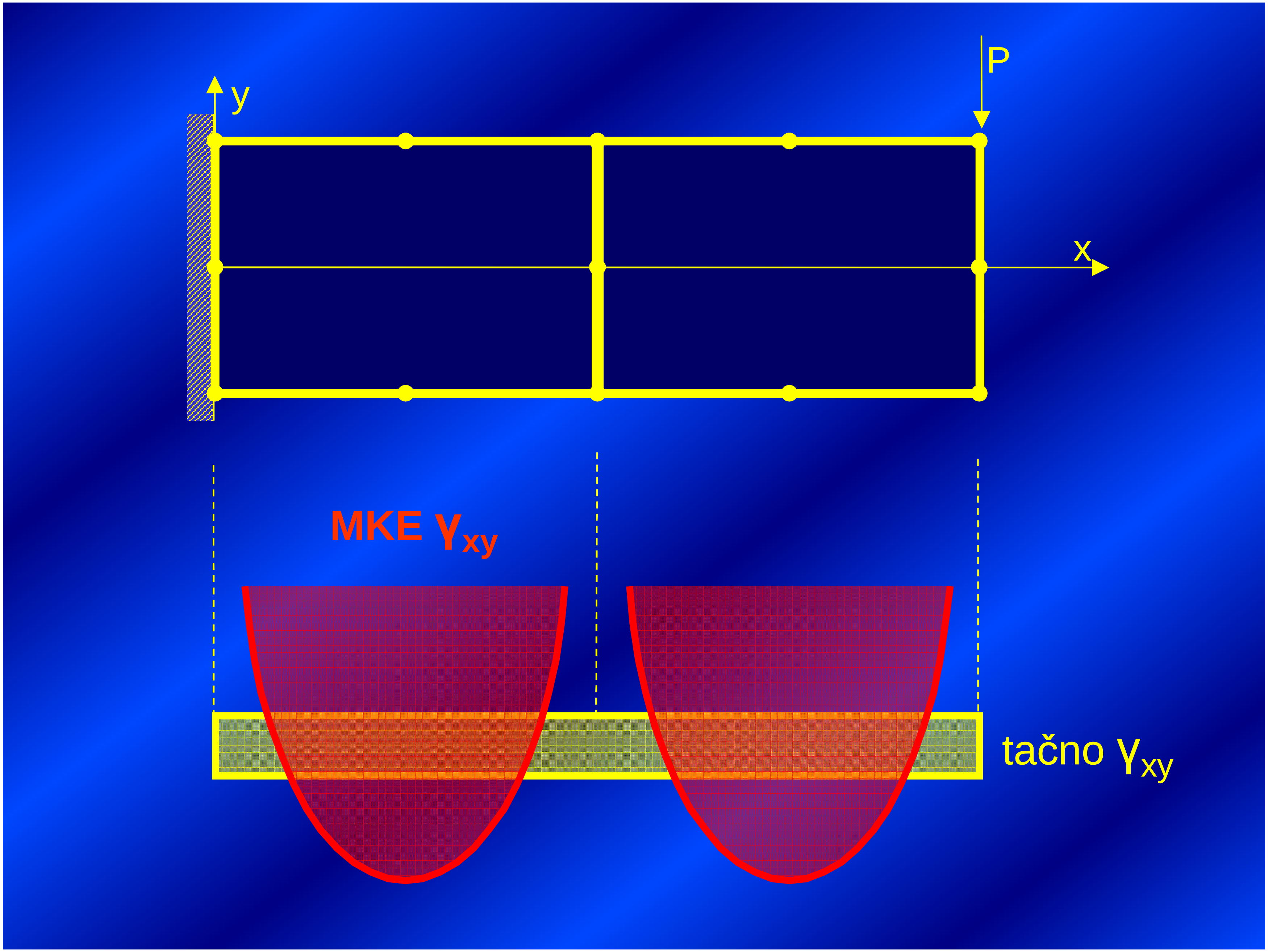
- greške diskretizacije
- greške aproksimacije
- greške implementacije u CAA softveru

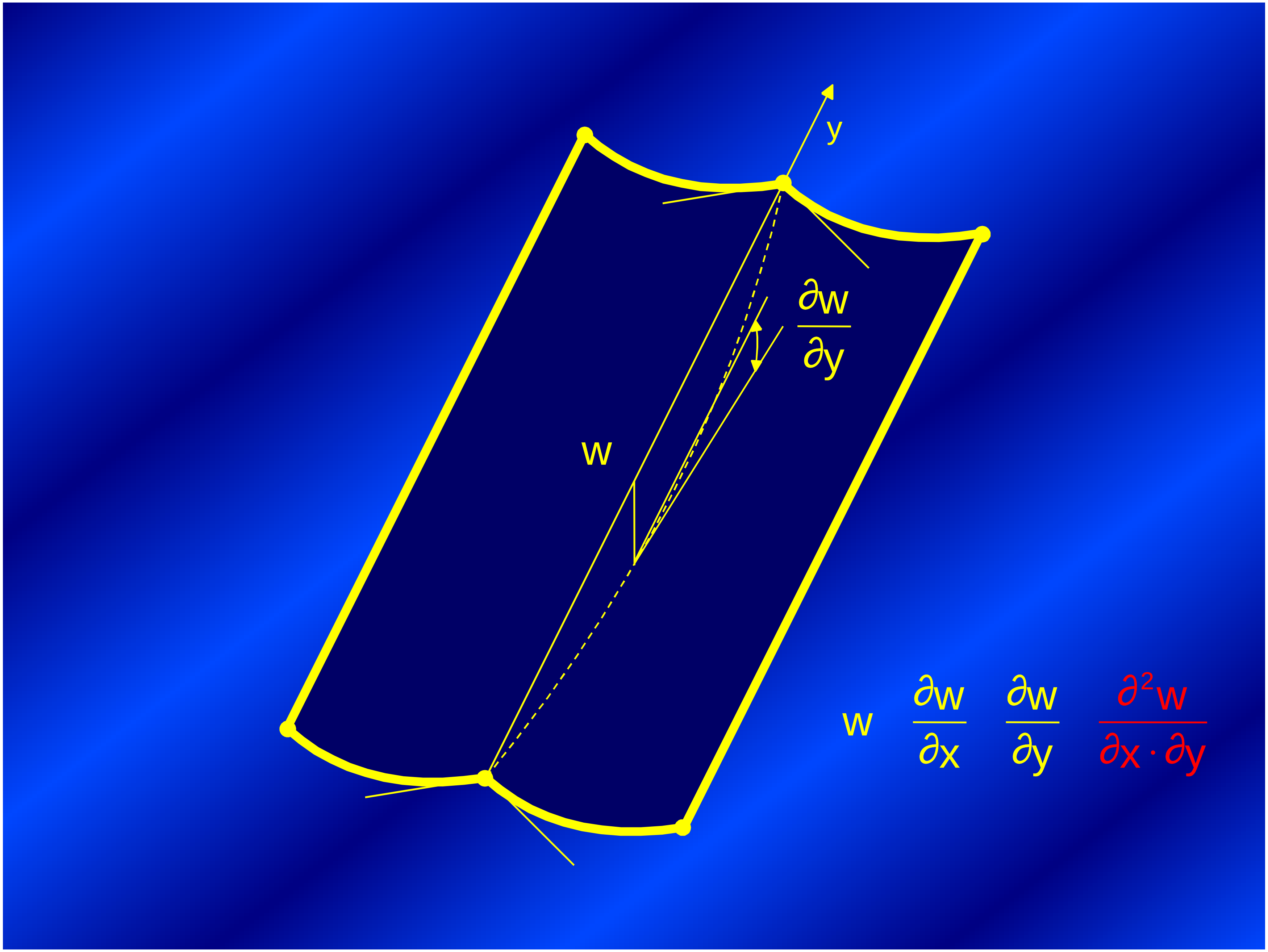

nestabilno
divergentno
gornja granica rešenja
"tačno" rešenje
konvergentno
donja granica rešenja
tačnost, stabilnost i konvergencija rešenja
broj konačnih elemenata, tačnost interpolacionih funkcija


metoda
sila
"tačno"
rešenje
metoda
deformacija
tačnost aproksimacije
pomeranja
broj konačnih elemenata,
tačnost interpolacionih funkcija


1
2
3
4
5
MKE M
tačno M


y
x
MKE γxy
tačno γxy

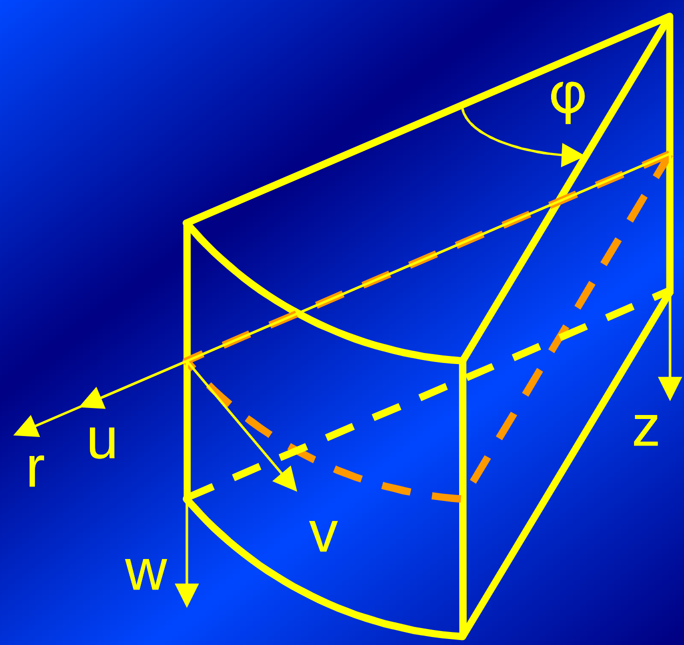

P
y
x
MKE γ_{xy}
tačno γ_{xy}


y
$\frac{\partial w}{\partial y}$
w
w $\frac{\partial w}{\partial x}$ $\frac{\partial w}{\partial y}$ $\frac{\partial^2 w}{\partial x \cdot \partial y}$

Kružna ploča

Komponentalna pomeranja i krivolinijske koordinate

- "u" - u pravcu radijusa ploče (r-koordinata)
- "v" - u pravcu tangente na koncentrični krug (φ-koordinata)
- "w" - u pravcu normale na srednju ravan ploče (z-koordinata)



Veze komponentalnih pomeranja i deformacija

$$\varepsilon_z = \frac{\partial w}{\partial z} \quad \Rightarrow \quad w = w(r,\varphi)$$

$$\gamma_{zr} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} = 0$$

$$\gamma_{z\varphi} = \frac{\partial v}{\partial z} + \frac{\partial w}{r \cdot \partial \varphi} = 0$$

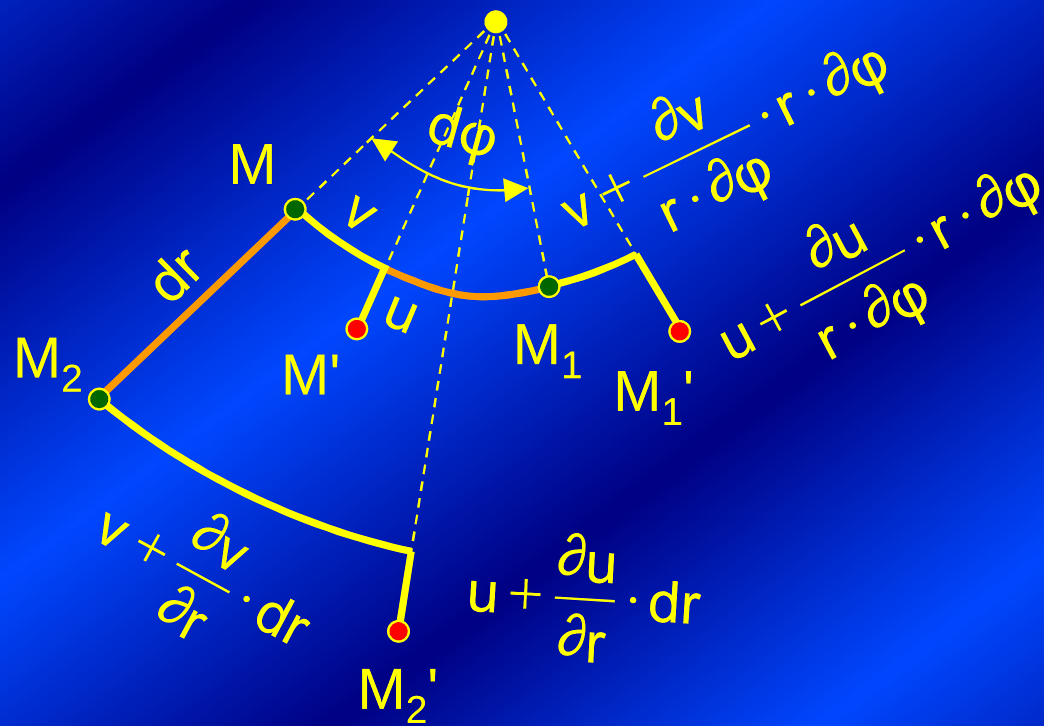
$$u = -z \cdot \frac{\partial w}{\partial r} \qquad v = -z \cdot \frac{\partial w}{r \cdot \partial \varphi}$$

pomeranja tačaka u ravni 0rφ:

$$M \longrightarrow M'$$

$$M_1 \longrightarrow M_1'$$

$$M_2 \longrightarrow M_2'$$



$$\varepsilon_r = \frac{\Delta dr}{dr} = \frac{u + \frac{\partial u}{\partial r} \cdot dr - u}{dr} = \frac{\partial u}{\partial r}$$

$$\varepsilon_\varphi = \varepsilon_\varphi^1 + \varepsilon_\varphi^2 = \frac{dv}{r \cdot \partial \varphi} + \frac{u}{r}$$

$$\varepsilon_\varphi^1 = \frac{v + \frac{\partial v}{r \cdot \partial \varphi} \cdot r \cdot \partial \varphi - v}{r \cdot \partial \varphi} = \frac{dv}{r \cdot \partial \varphi}$$

$$\varepsilon_\varphi^2 = \frac{(r+u) \cdot \partial \varphi - r \cdot \partial \varphi}{r \cdot \partial \varphi} = \frac{u}{r}$$

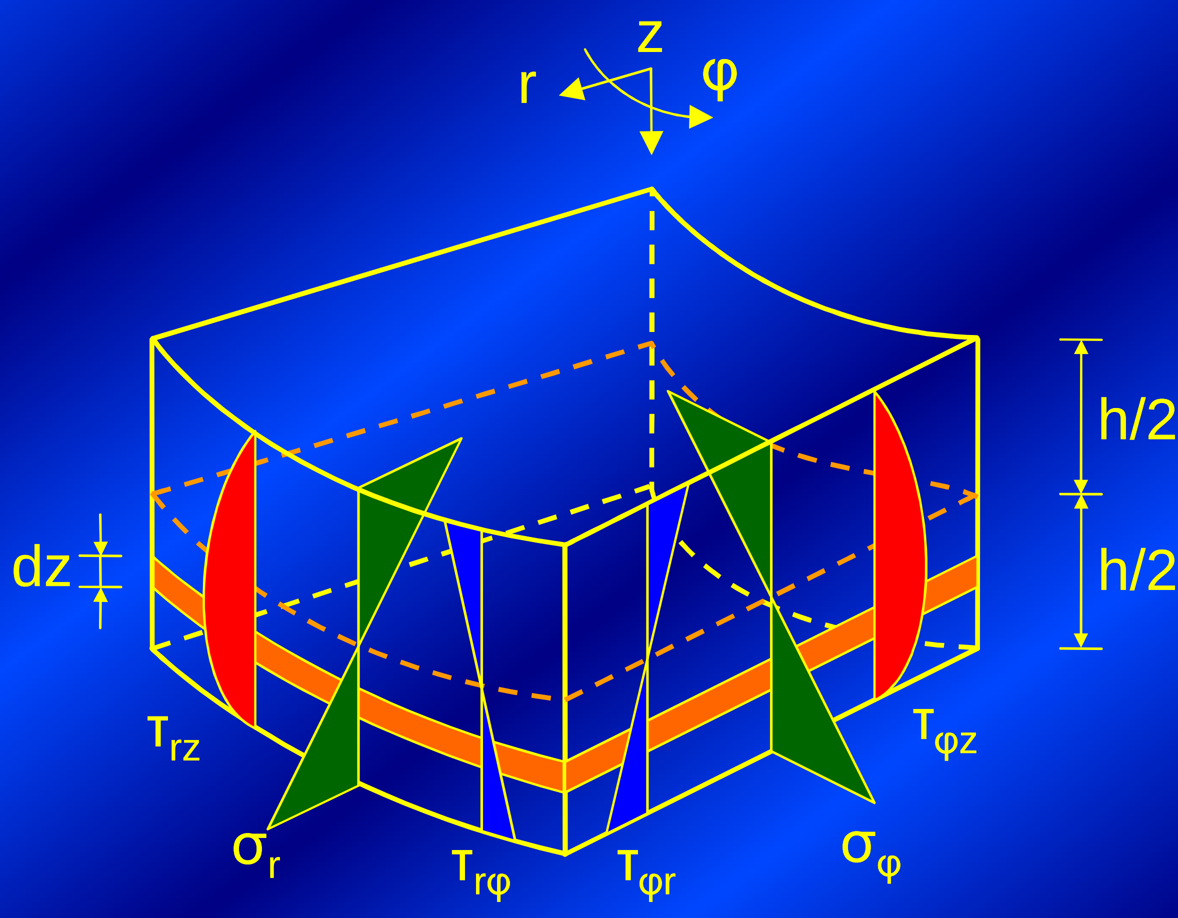
$$\gamma_{r\varphi} = \frac{v + \frac{\partial v}{\partial r}\cdot dr - \frac{v}{r}\cdot (r + dr)}{dr} + \frac{u + \frac{\partial u}{r\cdot \partial\varphi}\cdot r\cdot\partial\varphi - u}{r\cdot\partial\varphi} = \frac{\partial v}{\partial r} - \frac{v}{r} + \frac{\partial u}{r\cdot\partial\varphi}$$

$$\varepsilon_r = -z\cdot\frac{\partial^2 w}{\partial r^2}$$

$$\varepsilon_\varphi = -z\cdot\left[\frac{\partial}{r\cdot\partial\varphi}\left(\frac{\partial w}{r\cdot\partial\varphi}\right) + \frac{\partial w}{r\cdot\partial r}\right] = -z\cdot\left[\frac{\partial^2 w}{r^2\cdot\partial\varphi^2} + \frac{\partial w}{r\cdot\partial r}\right]$$

$$\gamma_{r\varphi} = -z\cdot\left[\frac{\partial}{\partial r}\left(\frac{\partial w}{r\cdot\partial\varphi}\right) - \frac{\partial w}{r^2\cdot\partial r} + \frac{\partial}{r\cdot\partial\varphi}\left(\frac{\partial w}{\partial r}\right)\right] = -2\cdot z\cdot\left[\frac{\partial^2 w}{r\cdot\partial r\cdot\partial\varphi} - \frac{\partial w}{r^2\cdot\partial\varphi}\right]$$

Komponentalni naponi



Veze komponentalnih napona, deformacija i sila u presecima

$$\sigma_r = \frac{E}{1-\nu^2}\cdot(\varepsilon_r + \nu\cdot\varepsilon_\varphi) \qquad \sigma_\varphi = \frac{E}{1-\nu^2}\cdot(\varepsilon_\varphi + \nu\cdot\varepsilon_r)$$

$$\tau_{r\varphi} = \frac{E}{2\cdot(1+\nu)}\cdot\gamma_{r\varphi} = G\cdot\gamma_{r\varphi}$$

$$M_x = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_r\cdot z\cdot dz \qquad M_\varphi = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_\varphi\cdot z\cdot dz \qquad M_{r\varphi} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \tau_{r\varphi}\cdot z\cdot dz$$

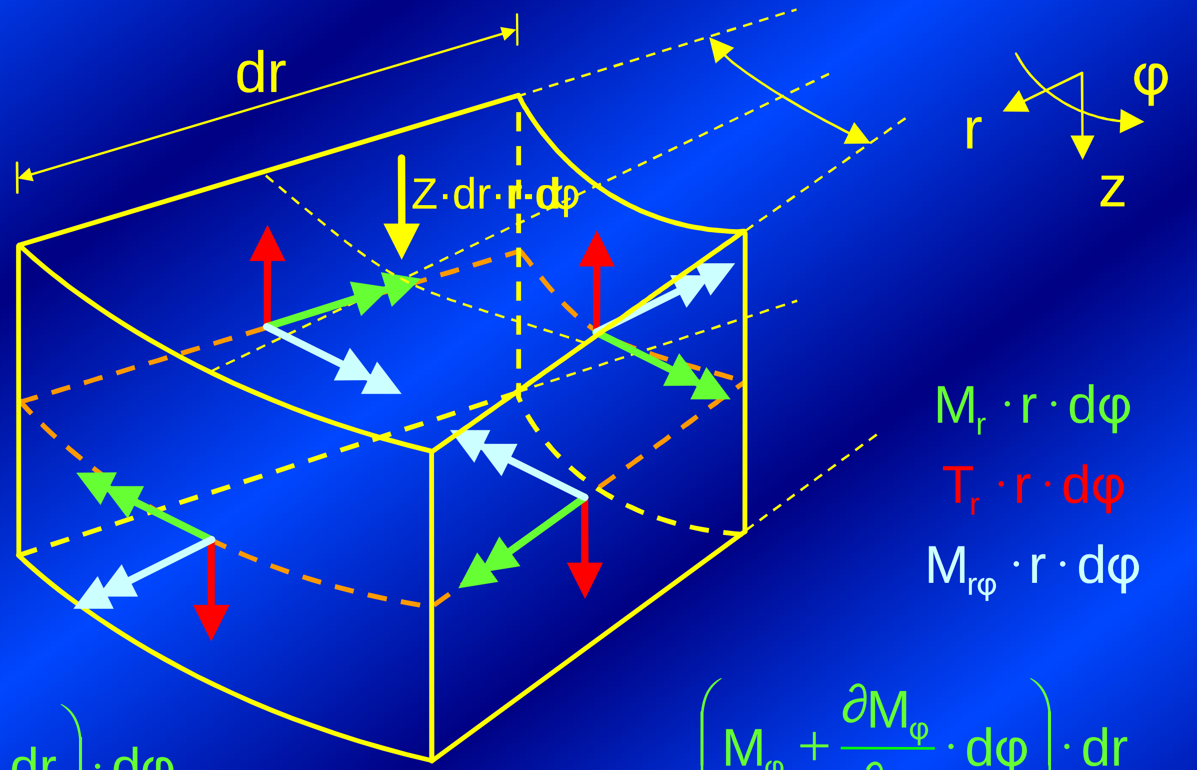
Veze komponentalnih pomeranja i sila u presecima

$$M_r = -\frac{E \cdot h^3}{12(1 - v^2)} \cdot \left[\frac{\partial^2 w}{\partial r^2} + v \cdot \left(\frac{\partial^2 w}{r^2 \cdot \partial \varphi^2} + \frac{\partial w}{r \cdot \partial r} \right) \right]$$

$$M_\varphi = -\frac{E \cdot h^3}{12(1 - v^2)} \cdot \left[\frac{\partial^2 w}{r^2 \cdot \partial \varphi^2} + \frac{\partial w}{r \cdot \partial r} + v \cdot \frac{\partial^2 w}{\partial r^2} \right]$$

$$M_{r\varphi} = -\frac{E \cdot h^3}{12(1 - v^2)} \cdot (1 - v) \cdot \left[\frac{\partial^2 w}{r \cdot \partial r \cdot \partial \varphi} - \frac{\partial w}{r^2 \cdot \partial r} \right]$$

Uslovi ravnoteže



$M_\varphi \cdot dr$

$T_\varphi \cdot dr$

$M_{r\varphi} \cdot dr$

$M_r \cdot r \cdot d\varphi$

$T_r \cdot r \cdot d\varphi$

$M_{r\varphi} \cdot r \cdot d\varphi$

$$\left(M_r \cdot r + \frac{\partial (M_r \cdot r)}{\partial r} \cdot dr \right) \cdot d\varphi$$

$$\left(T_r \cdot r + \frac{\partial (T_r \cdot r)}{\partial r} \cdot dr \right) \cdot d\varphi$$

$$\left(M_{r\varphi} \cdot r + \frac{\partial (M_{r\varphi} \cdot r)}{\partial r} \cdot dr \right) \cdot d\varphi$$

$$\left(M_\varphi + \frac{\partial M_\varphi}{\partial \varphi} \cdot d\varphi \right) \cdot dr$$

$$\left(T_\varphi + \frac{\partial T_\varphi}{\partial \varphi} \cdot d\varphi \right) \cdot dr$$

$$\left(M_{r\varphi} + \frac{\partial M_{r\varphi}}{\partial \varphi} \cdot d\varphi \right) \cdot dr$$

$$\sum M_{r=r+dr/2} = 0 \qquad \frac{\partial(M_r \cdot r)}{\partial r} + \frac{\partial M_{r\varphi}}{\partial \varphi} - M_\varphi - T_r \cdot r = 0$$

$$\sum M_{\varphi=d\varphi/2} = 0 \qquad \frac{\partial(M_{r\varphi} \cdot r)}{\partial r} + \frac{\partial M_\varphi}{\partial \varphi} + M_{r\varphi} - T_\varphi \cdot r = 0$$

$$\sum Z = 0 \qquad \frac{\partial(T_r \cdot r)}{\partial r} + \frac{\partial T_\varphi}{\partial \varphi} + Z \cdot r = 0$$

$$T_r = -k \cdot \frac{\partial}{\partial r}\left(\frac{\partial^2 w}{\partial r^2} + \frac{\partial w}{r \cdot \partial r} + \frac{\partial^2 w}{r^2 \cdot \partial \varphi^2}\right)$$

$$T_\varphi = -k \cdot \frac{\partial}{r \cdot \partial \varphi}\left(\frac{\partial^2 w}{\partial r^2} + \frac{\partial w}{r \cdot \partial r} + \frac{\partial^2 w}{r^2 \cdot \partial \varphi^2}\right)$$

Parcijalna diferencijalna jednačina savijanja kružne ploče

$$\left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{r \cdot \partial r} + \frac{\partial^2}{r^2 \cdot \partial \varphi^2}\right)\left(\frac{\partial^2 w}{\partial r^2} + \frac{\partial w}{r \cdot \partial r} + \frac{\partial^2 w}{r^2 \cdot \partial \varphi^2}\right) = \frac{Z}{k}$$

Rotaciona simetrija kružne ploče

$$\left(\frac{d^2}{dr^2}+\frac{d}{r\cdot dr}\right)\left(\frac{d^2w}{dr^2}+\frac{dw}{r\cdot dr}\right)=\frac{Z}{k}$$

$$\frac{d^4w}{dr^4}+\frac{2}{r}\cdot\frac{d^3w}{dr^3}-\frac{1}{r^2}\cdot\frac{d^2w}{dr^2}+\frac{1}{r^3}\cdot\frac{dw}{dr}=\frac{Z}{k}$$

$$w=w_h+w_p \qquad r=e^t \qquad \Rightarrow \qquad \frac{d^4w}{dt^4}-4\cdot\frac{d^3w}{dr^3}+4\cdot\frac{d^2w}{dr^2}=0$$

$$w_h=A+B\cdot t+C\cdot e^{2\cdot t}+D\cdot t\cdot e^{2\cdot t}$$

$$w_h=A+B\cdot \ln r+C\cdot r^2+D\cdot r^2\cdot \ln r$$

$$w = w_p + C_1 + C_2 \cdot \rho^2 + C_3 \cdot \rho^2 \cdot \ln\rho + C_4 \cdot \ln\rho \qquad \rho = \frac{r}{a}$$

$$M_r = -k\left(\frac{d^2w}{dr^2} + v \cdot \frac{dw}{r \cdot dr}\right)$$

$$M_\varphi = -k\left(\frac{dw}{r \cdot dr} + v \cdot \frac{d^2w}{dr^2}\right)$$

$$T_r = -k \cdot \frac{d}{dr}\left(\frac{d^2w}{dr^2} + v \cdot \frac{dw}{r \cdot dr}\right)$$

$$M_{r\varphi} = M_{\varphi r} = T_\varphi = 0$$

$$\frac{dw}{dr} = \left[\frac{dw_p}{d\rho} + 2 \cdot \rho \cdot C_2 + \rho \cdot C_3 \cdot (1 + 2 \cdot \ln\rho) + C_4 \cdot \frac{1}{\rho}\right] \qquad \rho = \frac{r}{a}$$

$$M_r = -\frac{k}{a^2} \cdot \left\{\frac{d^2 w_p}{d\rho^2} + \frac{v}{\rho} \cdot \frac{dw_p}{d\rho} + (1 + v) \cdot \left[2 \cdot C_2 + C_3 \cdot \left(\frac{3 + v}{1 + v} + 2 \cdot \ln\rho\right) - \right.\right.$$

$$\left.\left. - C_4 \cdot \frac{1 - v}{1 + v} \cdot \frac{1}{\rho^2}\right]\right\}$$

$$M_\varphi = -\frac{k}{a^2}\cdot\left\{v\cdot\frac{d^2w_p}{d\rho^2}+\frac{dw_p}{\rho\cdot d\rho}+(1+v)\cdot\left[2\cdot C_2+C_3\cdot\right.\right.$$

$$\left.\left.\cdot\left(\frac{1+3\cdot v}{1+v}+2\cdot\ln\rho\right)+C_4\cdot\frac{1-v}{1+v}\cdot\frac{1}{\rho^2}\right]\right\}$$

$$T_r = -\frac{k}{a^2}\cdot\left(\frac{d^3w_p}{d\rho^3}+\frac{d^2w_p}{\rho\cdot d\rho^2}-\frac{dw_p}{\rho^2\cdot d\rho}+\frac{4\cdot C_3}{\rho}\right)$$

Partikularni integral diferencijalne jednačine savijanja kružne ploče

$$\frac{d^2w}{dr^2}+\frac{dw}{r\cdot dr}=-\frac{M}{k}=\frac{d}{r\cdot dr}\cdot\left(\frac{dw}{r\cdot dr}\right) \qquad M=\frac{M_r+M_\varphi}{(1+v)}$$

$$\frac{d^2M}{dr^2}+\frac{dM}{r\cdot dr}=-Z=\frac{d}{r\cdot dr}\left(r\cdot\frac{dM}{dr}\right) \quad \Rightarrow \quad \frac{d}{dr}\left(r\cdot\frac{dM}{dr}\right)=-Z\cdot r$$

$$M=-\int_0^a\frac{dr}{r}\int_0^a Z\cdot r\cdot dr \qquad w_p=-\frac{1}{k}\cdot\int_0^a\frac{dr}{r}\int_0^a M\cdot r\cdot dr$$

$$M=-Z_0\cdot\int_0^a\frac{dr}{r}\int_0^a r\cdot dr=-\frac{Z_0\cdot r^2}{4} \qquad w_p=-\frac{Z_0}{4\cdot k}\cdot\int_0^a\frac{dr}{r}\int_0^a r^3\cdot dr=\frac{Z_0\cdot r^4}{64\cdot k}$$

Konturni uslovi savijanja kružne ploče

$$\rho \to 0 \quad w = w_p + C_1 + C_2 \cdot \rho^2 + C_3 \cdot \rho^2 \cdot \ln\rho + C_4 \cdot \ln\rho \to \infty \Rightarrow$$

$$\Rightarrow C_3 = C_4 = 0$$

$$\left.\begin{aligned} w &= 0 \\ M_r &= 0 \end{aligned}\right\}$$

slobodno oslonjena ivica

$$\left.\begin{aligned} w &= 0 \\ \frac{dw}{dr} &= 0 \end{aligned}\right\}$$

uklještena ivica

$$\left.\begin{aligned} T_r &= 0 \\ M_r &= 0 \end{aligned}\right\}$$

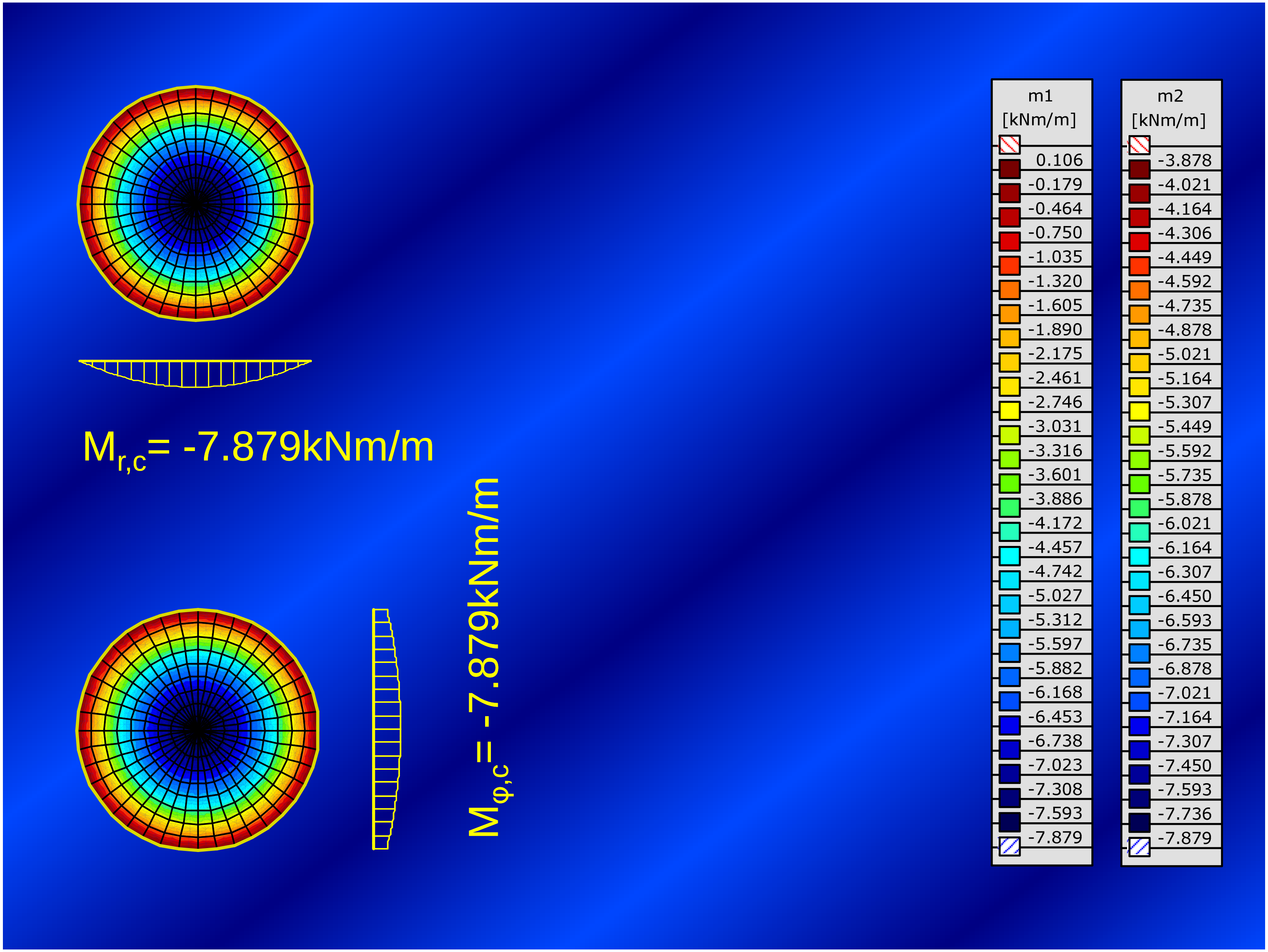
slobodna ivica

Slobodno oslonjena kružna ploča - jednakopodeljeno opterećenje

$$\left.\begin{aligned} w &= \frac{Z_0 \cdot a^4}{64 \cdot k} \cdot (\rho^4 + C_1 + C_2 \cdot \rho^2) = 0 \\ M_r &= -\frac{Z_0 \cdot a^2}{16}[(3+v)\cdot\rho^2 + 0.5\cdot(1+v)\cdot C_2] = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} C_1 &= 2\cdot\frac{3+v}{1+v} - 1 \\ C_2 &= -2\cdot\frac{3+v}{1+v} \end{aligned}$$

$$w = \frac{Z_0 \cdot a^4}{64 \cdot k} \cdot \left[(1-\rho^2)^2 + \frac{(1-\rho^2)}{(1+v)}\right] \qquad M_r = \frac{Z_0 \cdot a^2}{16} \cdot (3+v)\cdot(1-\rho^2)$$

$$M_\varphi = \frac{Z_0 \cdot a^2}{16} \cdot [(3+v) - \rho^2 \cdot (1+3\cdot v)] \qquad T_r = -\frac{Z_0 \cdot a}{2} \cdot \rho$$



$M_{r,c}$= -7.879kNm/m
$M_{\varphi,c}$= -7.879kNm/m
m1
[kNm/m]
0.106
-0.179
-0.464
-0.750
-1.035
-1.320
-1.605
-1.890
-2.175
-2.461
-2.746
-3.031
-3.316
-3.601
-3.886
-4.172
-4.457
-4.742
-5.027
-5.312
-5.597
-5.882
-6.168
-6.453
-6.738
-7.023
-7.308
-7.593
-7.879
m2
[kNm/m]
-3.878
-4.021
-4.164
-4.306
-4.449
-4.592
-4.735
-4.878
-5.021
-5.164
-5.307
-5.449
-5.592
-5.735
-5.878
-6.021
-6.164
-6.307
-6.450
-6.593
-6.735
-6.878
-7.021
-7.164
-7.307
-7.450
-7.593
-7.736
-7.879

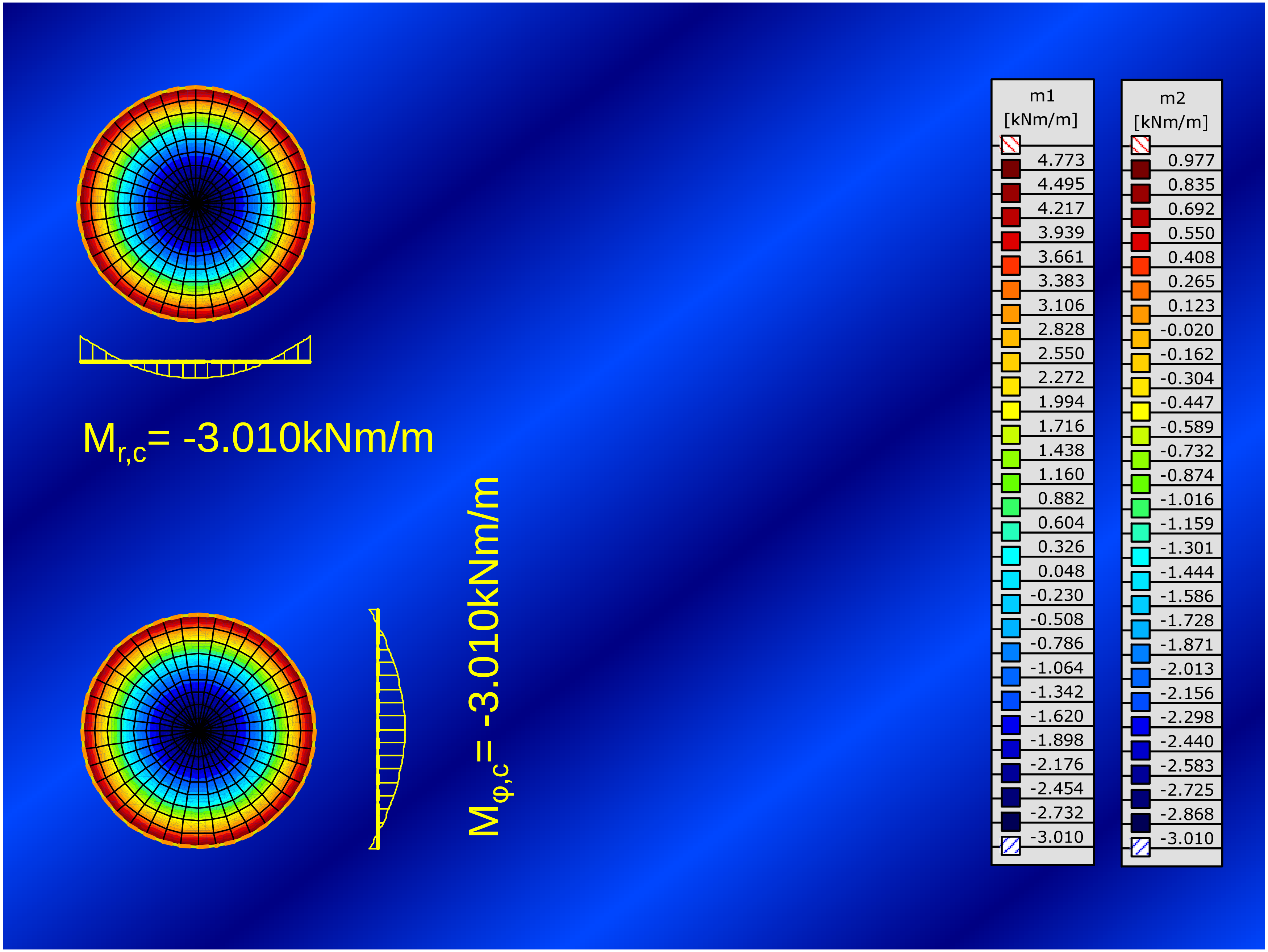
Uklještena kružna ploča - jednakopodeljeno opterećenje

$$\left.\begin{aligned} w &= 0 \\ \frac{dw}{dr} &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} C_1 &= 1 \\ C_2 &= -2 \end{aligned} \qquad w = \frac{Z_0 \cdot a^4}{64 \cdot k} \cdot (1 - \rho^2)^2$$

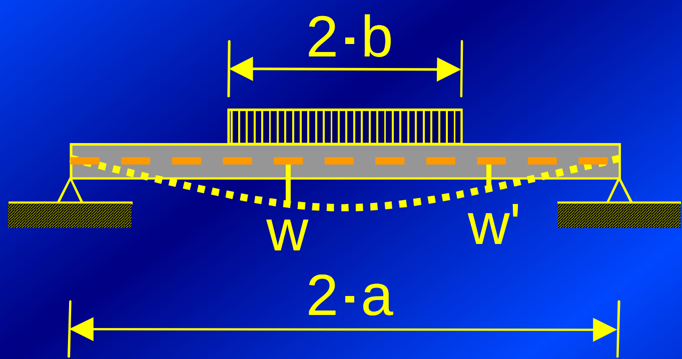
$$M_r = \frac{Z_0 \cdot a^2}{16} \cdot [(1+v) \cdot (3+v) \cdot \rho^2]$$

$$M_\varphi = \frac{Z_0 \cdot a^2}{16} \cdot [(1+v) - \rho^2 \cdot (1 + 3 \cdot v)]$$

$$T_r = -\frac{Z_0 \cdot a}{2} \cdot \rho \qquad A = \frac{R}{2 \cdot a \cdot \pi} = \frac{Z_0 \cdot a}{2}$$


$M_{r,c}$= -3.010kNm/m
$M_{\varphi,c}$= -3.010kNm/m
m1
[kNm/m]
4.773
4.495
4.217
3.939
3.661
3.383
3.106
2.828
2.550
2.272
1.994
1.716
1.438
1.160
0.882
0.604
0.326
0.048
-0.230
-0.508
-0.786
-1.064
-1.342
-1.620
-1.898
-2.176
-2.454
-2.732
-3.010
m2
[kNm/m]
0.977
0.835
0.692
0.550
0.408
0.265
0.123
-0.020
-0.162
-0.304
-0.447
-0.589
-0.732
-0.874
-1.016
-1.159
-1.301
-1.444
-1.586
-1.728
-1.871
-2.013
-2.156
-2.298
-2.440
-2.583
-2.725
-2.868
-3.010

Slobodno oslonjena kružna ploča - delimično jednakopodeljeno opterećenje



$$w = \frac{Z_0 \cdot a^4}{64 \cdot k} \cdot \left(\rho^4 + C_1 + C_2 \cdot \rho^2\right)$$

$$w' = C'_1 + C'_2 \cdot \rho^2 + C'_3 \cdot \rho^2 \cdot \ln\rho + C'_4 \cdot \ln\rho$$

$$\rho = 0 \qquad w' = 0$$

$$\rho = \frac{b}{a} = \beta \qquad T_r = T'_r \Rightarrow \frac{d^3w}{dr^3} = \frac{d^3w'}{dr^3}$$

$$M_r = 0 \qquad \frac{dw}{dr} = \frac{dw'}{dr} \qquad M_r = M'_r \Rightarrow \frac{d^2w}{dr^2} = \frac{d^2w'}{dr^2}$$

$$0 \le \rho \le \beta \qquad w = \frac{Z_0 \cdot a^4 \cdot \beta^2}{16 \cdot k} \cdot \left\{ \frac{\rho^4}{4 \cdot \beta^2} + \rho^3 \cdot \left[\frac{(1-v) \cdot \beta^2 - 4}{2 \cdot (1+v)} + 2 \cdot \ln\beta \right] + \right.$$

$$\left. + \frac{4 \cdot (3+v) - (7 + 3 \cdot v) \cdot \beta^2 - 4}{4 \cdot (1+v)} + \beta^2 \cdot \ln\beta \right\}$$

$$M_r = \frac{Z_0 \cdot a^2 \cdot \beta^2}{4} \cdot \left[-(1+v) \cdot \ln\beta + 1 - \frac{(1-v)}{4} \cdot \beta^2 - \frac{(1-v)}{4 \cdot \beta^2} \cdot \rho^2 \right]$$

$$M_\varphi = \frac{Z_0 \cdot a^2 \cdot \beta^2}{4} \cdot \left[-(1+v) \cdot \ln\beta + 1 - \frac{(1-v)}{4} \cdot \beta^2 - \frac{(1+3 \cdot v)}{4 \cdot \beta^2} \cdot \rho^2 \right]$$

$$T_r = -\frac{Z_0 \cdot a \cdot \rho}{2}$$

$$\beta \le \rho \le 1 \qquad w' = \frac{Z_0 \cdot a^4 \cdot \beta^2}{16 \cdot k} \cdot \left\{ \frac{2 \cdot (3+v) - (1-v) \cdot \beta^2}{2 \cdot (1+v)} \cdot \right.$$

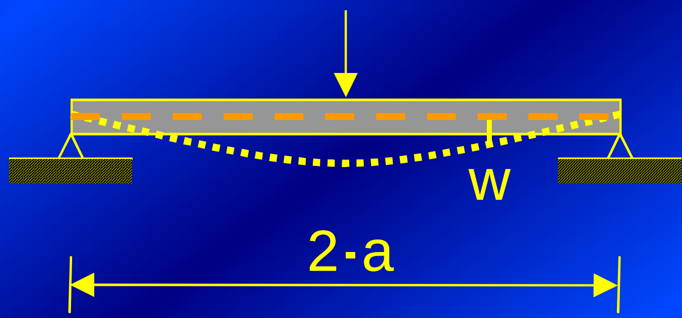
$$\left. \cdot (1-\rho^2) + 2 \cdot \rho^2 \cdot \ln\rho + \beta^2 \cdot \ln\beta \right\}$$

$$M'_r = \frac{Z_0 \cdot a^2 \cdot \beta^2}{4} \cdot \left[-(1+v) \cdot \ln\beta + 1 - \frac{(1-v)}{4} \cdot \beta^2 \left(\frac{1}{\rho^2} - 1 \right) \right]$$

$$M'_\varphi = \frac{Z_0 \cdot a^2 \cdot \beta^2}{4} \cdot \left[-(1+v) \cdot \ln\beta + (1-v) - \frac{(1-v)}{4} \cdot \beta^2 \left(\frac{1}{\rho^2} + 1 \right) \right]$$

$$T'_r = -\frac{Z_0 \cdot a \cdot \beta^2}{2 \cdot \rho}$$

Slobodno oslonjena kružna ploča - koncentrisana sila u centru



$$P = Z_0 \cdot \pi \cdot a^2 \cdot \beta^2$$

$$\beta \to 0$$

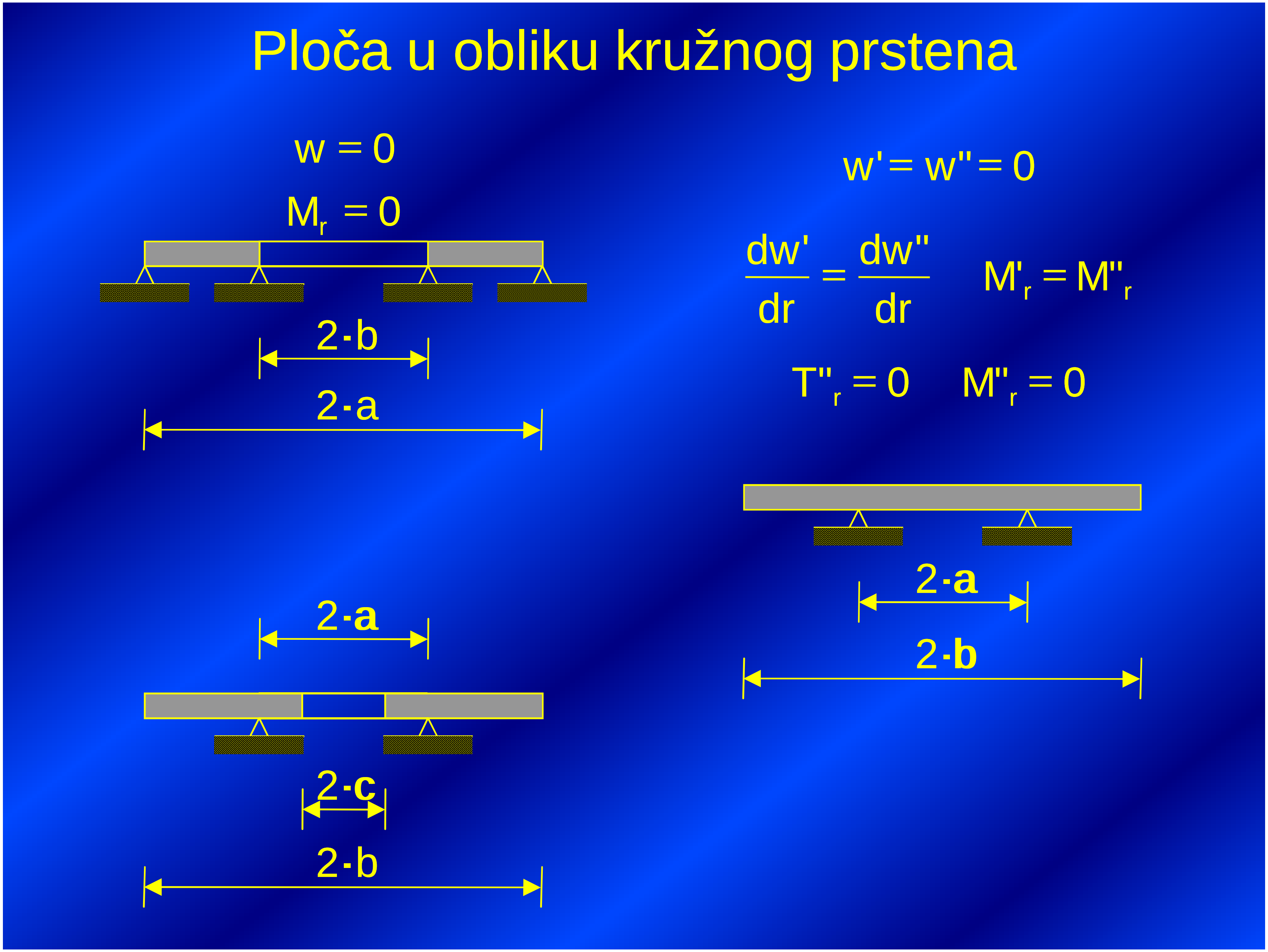
$$w = \frac{P \cdot a^2}{16 \cdot \pi \cdot k} \cdot \left[\frac{3+\nu}{1+\nu} \cdot (1-\rho^2) + 2 \cdot \rho^2 \cdot \ln\rho \right]$$

$$w_{max} = \frac{P \cdot a^2}{k \cdot \pi} \cdot \frac{3+\nu}{16 \cdot (1+\nu)}$$

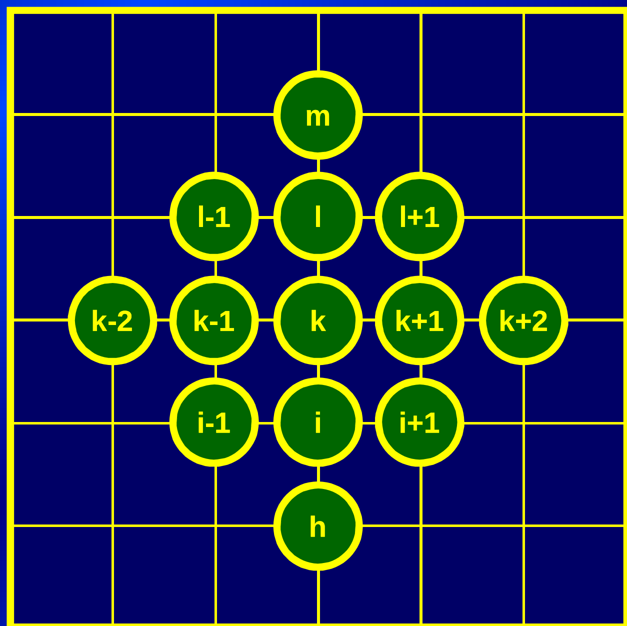
$$M_r = -\frac{P \cdot (1+\nu)}{4 \cdot \pi} \cdot \ln\rho$$

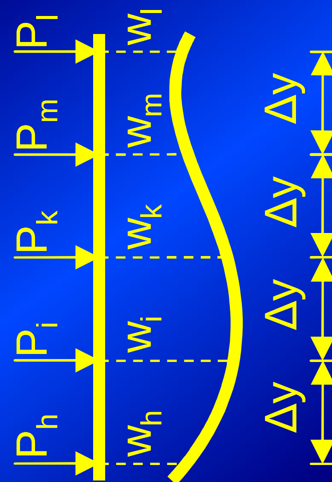
$$T_r = -\frac{P}{2 \cdot \pi \cdot a \cdot \rho}$$

$$M_r = -\frac{P}{4 \cdot \pi} \cdot [(1-\nu) - (1+\nu) \cdot \ln\rho]$$


Ploča u obliku kružnog prstena
w = 0
M_r = 0
2·b
2·a
w' = w'' = 0
dw'/dr = dw''/dr
M'_r = M''_r
T''_r = 0
M''_r = 0
2·a
2·b
2·a
2·c
2·b

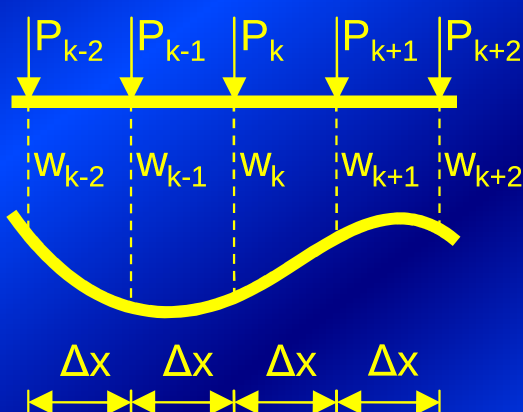
Metoda konačnih razlika - diferencncna metoda





$$\left(\frac{\partial w}{\partial x}\right)_k = \frac{w_{k+1} - w_{k-1}}{2 \cdot \Delta x}$$

$$\left(\frac{\partial w}{\partial y}\right)_k = \frac{w_l - w_i}{2 \cdot \Delta y}$$



$$\left(\frac{\partial^2 w}{\partial x^2}\right)_k = \frac{w_{k+1} - 2 \cdot w_k + w_{k-1}}{\Delta x^2}$$

$$\left(\frac{\partial^2 w}{\partial y^2}\right)_k = \frac{w_l - 2 \cdot w_k + w_i}{\Delta y^2}$$

$$\left(\frac{\partial^2 w}{\partial x \cdot \partial y}\right)_k = \frac{\left(\frac{\partial w}{\partial x}\right)_l - \left(\frac{\partial w}{\partial x}\right)_i}{2 \cdot \Delta x} = \frac{w_{l+1} - w_{l-1} + w_{i+1} - w_{i-1}}{4 \cdot \Delta x \cdot \Delta y}$$

$$\left(\frac{\partial^3 w}{\partial x^3}\right)_k = \frac{w_{k+2} - 2 \cdot w_{k+1} + 2 \cdot w_{k-1} - w_{k-2}}{2 \cdot \Delta x^3}$$

$$\left(\frac{\partial^3 w}{\partial y^3}\right)_k = \frac{w_m - 2 \cdot w_l + 2 \cdot w_i - w_h}{2 \cdot \Delta y^3}$$

$$\left(\frac{\partial^4 w}{\partial x^4}\right)_k = \frac{w_{k+2} - 4 \cdot w_{k+1} + 6 \cdot w_k - 4 \cdot w_{k-1} + w_{k-2}}{\Delta x^4}$$

$$\left(\frac{\partial^4 w}{\partial y^4}\right)_k = \frac{w_m - 4 \cdot w_l + 6 \cdot w_k - 4 \cdot w_i + w_h}{\Delta y^4}$$

$$\left(\frac{\partial^4 w}{\partial x^2 \cdot \partial y^2}\right)_k = \frac{4 \cdot w_k - 2 \cdot (w_l + w_i - w_{k+1} + w_{k-1})}{\Delta x^2 \cdot \Delta y^2} +$$

$$+ \frac{(w_{l+1} + w_{l-1} - w_{i+1} + w_{i-1})}{\Delta x^2 \cdot \Delta y^2}$$

$$\left(\frac{\partial^3 w}{\partial x^2 \cdot \partial y}\right)_k = \frac{w_{l+1} - 2 \cdot w_l + w_{l-1} - w_{i+1} + 2 \cdot w_i + w_{i-1}}{2 \cdot \Delta x^2 \cdot \Delta y}$$

$$\left(\frac{\partial^3 w}{\partial x \cdot \partial y^2}\right)_k = \frac{w_{l+1} - 2 \cdot w_{k+1} + w_{i+1} - \cdot w_{l-1} + 2 \cdot w_{k-1} + w_{i-1}}{2 \cdot \Delta x \cdot \Delta y^2}$$

Diferencna jednačina pravougaone ploče

$$w_k \cdot \left[6 \cdot \left(1+\frac{1}{\alpha^2}\right)+8\right]-4 \cdot \left[(1+\alpha^2)\cdot(w_{k+1}-w_{k-1})+\left(1+\frac{1}{\alpha^2}\right)\cdot(w_l+w_i)\right]+$$

$$+2\cdot(w_{i-1}+w_{l-1}+w_{i+1}+w_{l+1})+\alpha^2\cdot(w_{k+2}+w_{k-2})+$$

$$+\frac{1}{\alpha^2}\cdot(w_m+w_h)=\frac{Z_k\cdot\alpha^2\cdot\Delta x^4}{k} \qquad \alpha=\frac{\Delta x}{\Delta y}$$

$$\alpha=1 \qquad 20\cdot w_k-8\cdot(w_{k+1}+w_{k-1}+w_l+w_i)+$$

$$+2\cdot(w_{i-1}+w_{l-1}+w_{i+1}+w_{l+1})+$$

$$+(w_{k+2}+w_{k-2}+w_m+w_h)=\frac{Z_k\cdot\alpha^2\cdot\Delta x^4}{k}$$

Sile u presecima pravougaone ploče

$$M_{x,k} = \frac{k}{\Delta x^2} \cdot [-w_{k+1} - 2 \cdot w_k + w_{k-1} + \frac{v}{\alpha^2} \cdot (-w_l + 2 \cdot w_k - w_i)]$$

$$M_{y,k} = \frac{k}{\Delta y^2} \cdot [-w_l + 2 \cdot w_k - w_i + v \cdot \alpha^2 \cdot (-w_{k+1} - 2 \cdot w_k + w_{k-1})]$$

$$M_{xy,k} = \frac{k}{4 \cdot \Delta x^2 \cdot \alpha} \cdot (1 - v) \cdot [-w_{l+1} + w_{l-1} + w_{i+1} - w_{i-1})]$$

$$M_k = \frac{k}{\Delta x^2} \cdot [-w_{k+1} + 2 \cdot (1 + \alpha^2) \cdot w_k - w_{k-1} - w_i - w_l)]$$

$$T_{x,k} = \frac{k}{2 \cdot \Delta x^3} \cdot [w_{k-2} + (w_{i-1} + w_{l-1}) - (w_{i+1} + w_{l+1}) - $$
$$- w_{k+2} - 2 \cdot (1 + \alpha^2) \cdot (w_{k+1} + w_{k-1})]$$

$$T_{y,k} = \frac{k}{2 \cdot \alpha \cdot \Delta x^3} \cdot [w_h + (w_{i+1} + w_{i-1}) - (w_{l+1} + w_{l-1}) - $$
$$- w_m - 2 \cdot (1 + \alpha^2) \cdot (w_l + w_i)]$$

$$\overline{T}_{x,k} = \frac{k}{2 \cdot \Delta x^3} \cdot \left[w_{k-2} + (w_{k+1} + w_{k-1}) \cdot 2 \cdot \left(\frac{2 - v}{\alpha^2} + 1 \right) + \right.$$
$$\left. + \frac{2 - v}{\alpha^2} \cdot (w_{i-1} + w_{i+1} - w_{l-1} - w_{l+1}) - w_{k+2} \right]$$

$$\overline{T}_{y,k} = \frac{k}{2 \cdot \alpha \cdot \Delta x^3} \cdot \{w_h + (w_l + w_i) \cdot [2 + 2 \cdot \alpha^2 \cdot (2 - v)] + $$
$$+ \alpha^2 (2 - v) \cdot (w_{i+1} + w_{i-1} - w_{l+1} - w_{l-1}) - w_m]$$

Konturni uslovi (kontura sa normalom u pravcu y-ose)

$$\left.\begin{array}{c} w = 0 \\ \dfrac{\partial w}{\partial y} = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{c} w_\kappa = 0 \\ \dfrac{w_{\kappa+1} - w_{k-1}}{2 \cdot \Delta y} = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{c} w_\kappa = 0 \\ w_{\kappa+1} = w_{k-1} \end{array}\right\}$$

uklještena ivica

$$\left.\begin{array}{c} w = 0 \\ \dfrac{\partial^2 w}{\partial y^2} = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{c} w_\kappa = 0 \\ \dfrac{w_{\kappa+1} - 2 \cdot w_k + w_{k-1}}{\Delta y^2} = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{c} w_\kappa = 0 \\ w_{\kappa+1} = -w_{k-1} \end{array}\right\}$$

slobodno oslonjena ivica

$$\left.\begin{array}{c} M_y = 0 \\ \overline{T}_y = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{c} w_l = 2 \cdot w_k + v \cdot \alpha^2 \cdot (-w_{\kappa+1} + 2 \cdot w_k - w_{k-1}) \\ w_m = w_h + [2 + 2 \cdot \alpha^2 \cdot (2 + v) \cdot (w_l - w_i) + \\ + \alpha^2 \cdot (2 - v) \cdot (w_{i+1} - w_{i-1} - w_{l+1} - w_{l-1}) \end{array}\right\}$$

slobodna ivica

Tok proračuna ploča primenom diferencnog postupka

- formiranje ortogonalne mreže tačaka "k" ploče sa odstojanjima "Δx" i "Δy"
- formiranje sistema diferencnih jednačina za sve tačke ploče
- određivanje nepoznatih pomeranja "w_k" za svaku tačku mreže
- određivanje sila u presecima za svaku tačku mreže

PLOČE OPTEREĆENE U SOPSTVENOJ RAVNI

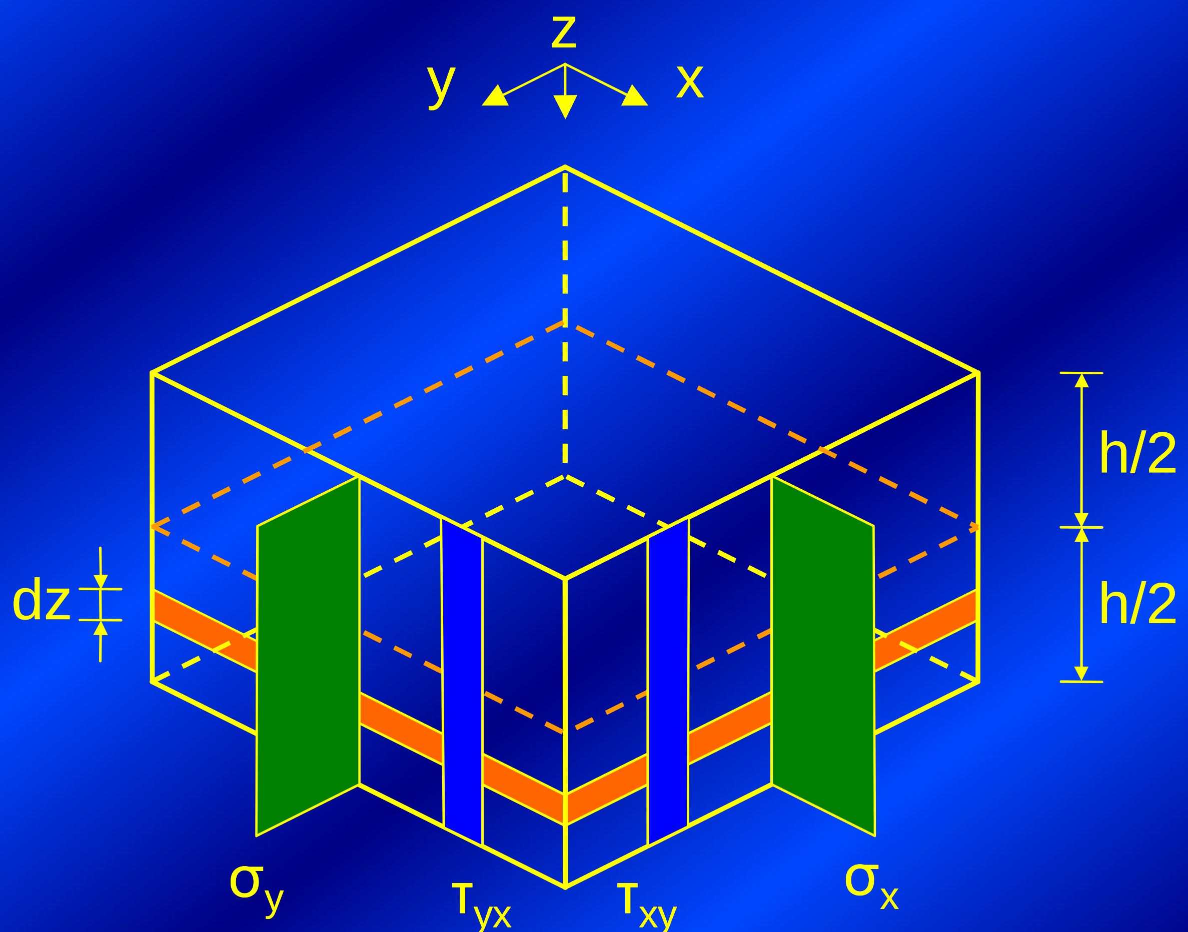
Osnovni pojmovi 1

- ploče - tela ograničena dvema (paralelnim) ravnima i cilindričnom površinom ortogonalnom na njih
- debljina ploče - "h" - rastojanje (paralelnih) ravni
- srednja ravan ploče - ravan koja polovi debljinu ploče
- kontura ploče - kriva preseka srednje ravni i cilindrične površine koja ograničava ploču

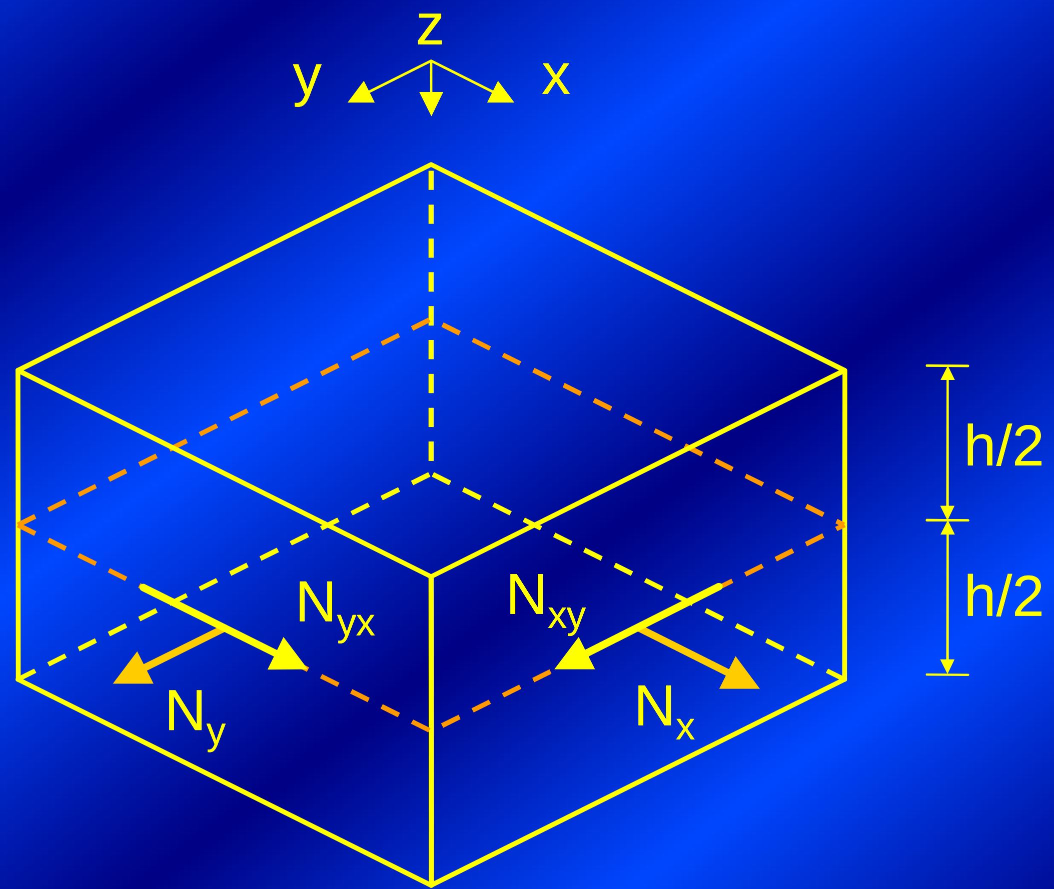
Osnovni pojmovi 2

- ploče opterećene površinskim i zapreminskim silama ravnomerno po debljini i paralelno srednjoj površi su opterećene u svojoj ravni
- za ovo tzv. ravno stanje napona srednja ravan ploče ostaje ravna i posle deformacije (nema krivljenja površi)

Komponentalni naponi



Sile u presecima



Sile u presecima i komponentalni naponi

$$N_x = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_x \cdot dz = \sigma_x \cdot h \qquad N_y = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_y \cdot dz = \sigma_y \cdot h$$

$$N_{xy} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \tau_{xy} \cdot dz = \tau_{xy} \cdot h \qquad N_{yx} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \tau_{yx} \cdot dz = \tau_{yx} \cdot h$$

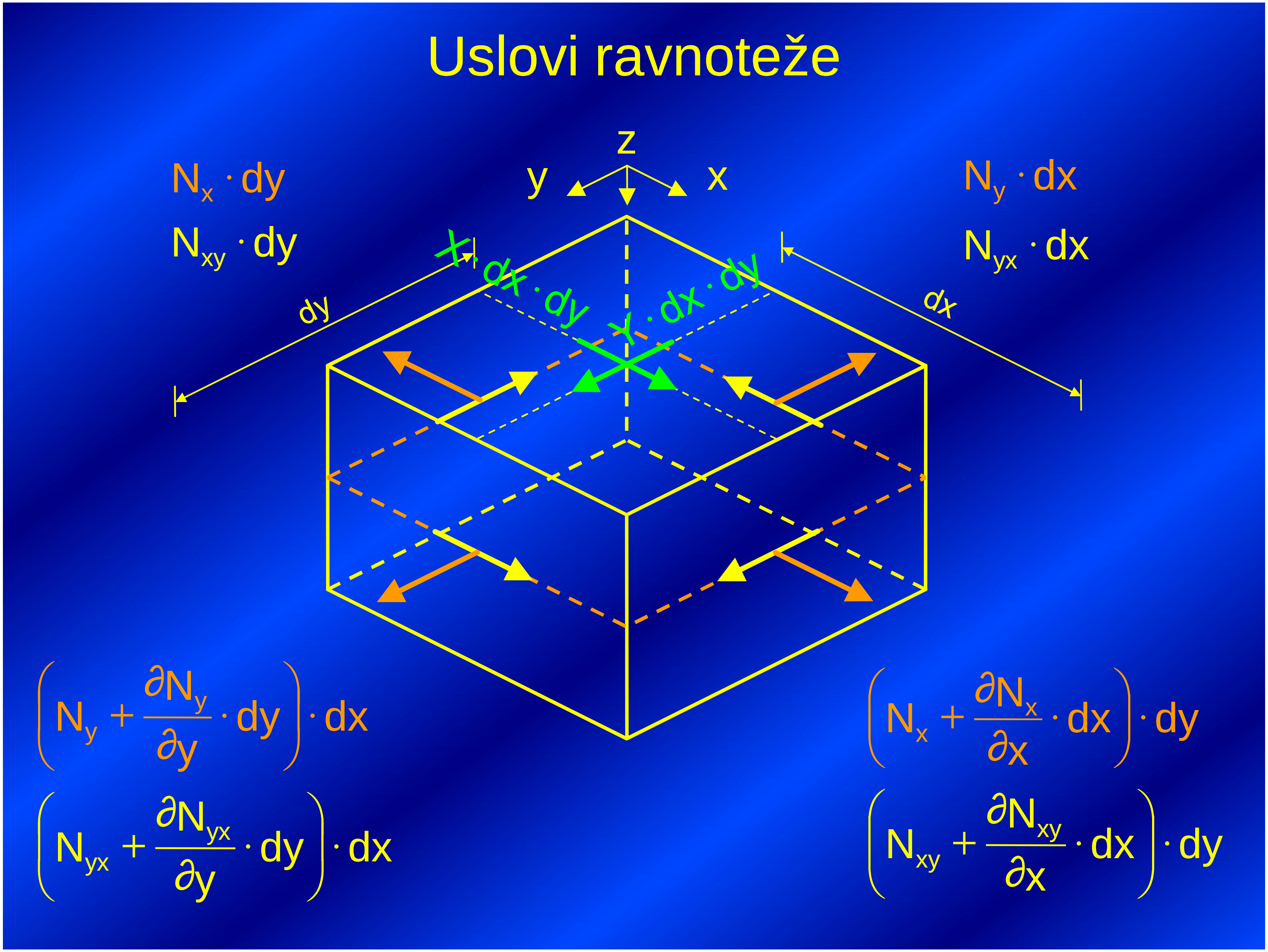
Veza zapreminskih, površinskih i linijskih opterećenja

$$F = h \cdot \overline{F}$$

$$X = h \cdot \overline{X} \qquad Y = h \cdot \overline{Y}$$

$$p_n = h \cdot \overline{p}_n$$

$$p_{nx} = h \cdot \overline{p}_{nx} \qquad p_{ny} = h \cdot \overline{p}_{ny}$$


Uslovi ravnoteže
z
y
x
$N_x \cdot dy$
$N_{xy} \cdot dy$
$N_y \cdot dx$
$N_{yx} \cdot dx$
dy
dx
$X \cdot dx \cdot dy$
$Y \cdot dx \cdot dy$
$\left(N_y + \frac{\partial N_y}{\partial y} \cdot dy\right) \cdot dx$
$\left(N_{yx} + \frac{\partial N_{yx}}{\partial y} \cdot dy\right) \cdot dx$
$\left(N_x + \frac{\partial N_x}{\partial x} \cdot dx\right) \cdot dy$
$\left(N_{xy} + \frac{\partial N_{xy}}{\partial x} \cdot dx\right) \cdot dy$

$$\sum X = 0 \qquad \Rightarrow \qquad \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + X = 0$$

$$\sum Y = 0 \qquad \Rightarrow \qquad \frac{\partial N_y}{\partial y} + \frac{\partial N_{yx}}{\partial x} + Y = 0$$

$$\sum M_x \equiv 0 \qquad \sum M_y \equiv 0$$

$$\sum M_z = 0 \qquad \Rightarrow \qquad N_{xy} = N_{yx}$$

Komponentalni naponi

$$\varepsilon_x = \frac{1}{E}\cdot(\sigma_x - v\cdot\sigma_y) \qquad \varepsilon_y = \frac{1}{E}\cdot(\sigma_y - v\cdot\sigma_x)$$

$$\gamma_{xy} = \frac{1}{G}\cdot\tau_{xy}$$

$$\sigma_x = \frac{E}{1-v^2}\cdot(\varepsilon_x + v\cdot\varepsilon_y) \qquad \sigma_y = \frac{E}{1-v^2}\cdot(\varepsilon_y + v\cdot\varepsilon_x)$$

$$\tau_{xy} = G\cdot\gamma_{xy}$$

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\underbrace{\frac{\partial^3 u}{\partial x \cdot \partial y^2} + \frac{\partial^3 v}{\partial x^2 \cdot \partial y} = \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} \qquad \frac{\partial^3 u}{\partial x \cdot \partial y^2} + \frac{\partial^3 v}{\partial x^2 \cdot \partial y} = \frac{\partial^2 \gamma_{xy}}{\partial x \cdot \partial y}}$$

$$\Downarrow$$

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \cdot \partial y}$$

$$N_x = \frac{E \cdot h}{1 - v^2} \cdot \left[\frac{\partial u}{\partial x} + v \cdot \frac{\partial v}{\partial y} \right] \qquad N_y = \frac{E \cdot h}{1 - v^2} \cdot \left[\frac{\partial v}{\partial y} + v \cdot \frac{\partial u}{\partial x} \right]$$

$$N_{xy} = \frac{E \cdot h}{2 \cdot (1 + v)} \cdot \left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right]$$

Diferencijalna jednačina ploče

Naponska funkcija (Airy, 1862)

$$
\left.
\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} + \frac{1-v}{2}\cdot\frac{\partial^2 u}{\partial y^2} + \frac{1+v}{2}\cdot\frac{\partial^2 v}{\partial x\cdot\partial y} + \frac{X}{D} = 0 \\[2ex]
&\frac{1+v}{2}\cdot\frac{\partial^2 u}{\partial x\cdot\partial y} + \frac{1-v}{2}\cdot\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{Y}{D} = 0
\end{aligned}
\right\}\ \text{metoda deformacija}
$$

$$
\left.
\begin{aligned}
&\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + X = 0 \qquad \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} + Y = 0 \\[2ex]
&\frac{\partial^2 u}{\partial x^2}(N_y - v\cdot N_x) + \frac{\partial^2 u}{\partial y^2}(N_x - v\cdot N_y) = 2\cdot(1+v)\cdot\frac{\partial^2 N_{xy}}{\partial x\cdot\partial y}
\end{aligned}
\right\}\ \text{metoda sila}
$$

$$\frac{\partial^2 F}{\partial x \cdot \partial y} = -N_{xy} \qquad X = -\frac{\partial U}{\partial x} \qquad Y = -\frac{\partial U}{\partial y}$$

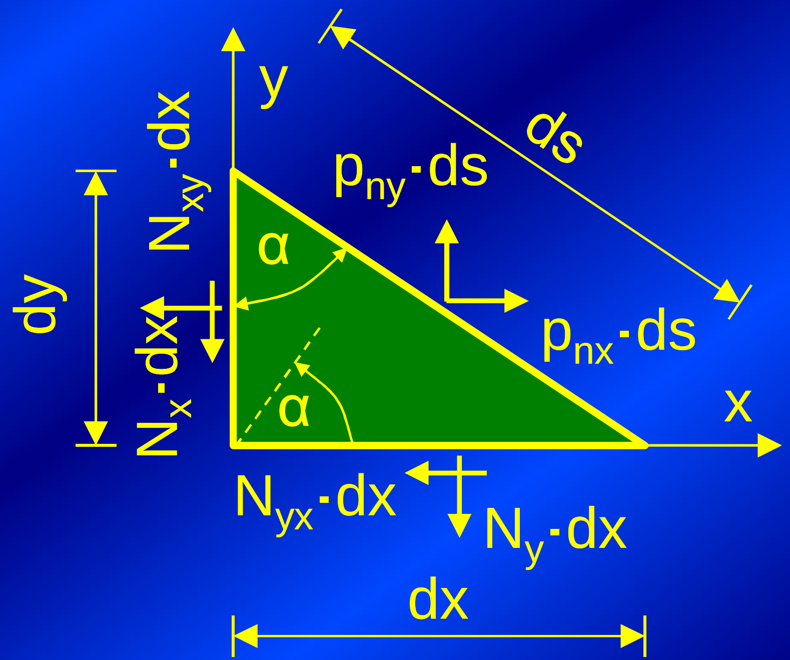
$$\frac{\partial}{\partial x}\left(N_x - \frac{\partial^2 F}{\partial y^2} - U\right) = 0 \qquad \frac{\partial}{\partial y}\left(N_y - \frac{\partial^2 F}{\partial x^2} - U\right) = 0$$

$$N_x = \frac{\partial^2 F}{\partial y^2} + U \qquad N_y = \frac{\partial^2 F}{\partial x^2} + U$$

$$\frac{\partial^4 F}{\partial x^4} + 2 \cdot \frac{\partial^4 F}{\partial x^2 \cdot \partial y^2} + \frac{\partial^4 F}{\partial y^4} + (1 - v) \cdot \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) = 0$$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \qquad \Delta\Delta F + (1 - v) \cdot \Delta U = 0 \qquad \Delta\Delta F = 0$$

Konturni uslovi



$$\cos\alpha = \frac{dy}{ds} \qquad \sin\alpha = -\frac{dx}{ds}$$

- uslovi po silama
- uslovi po pomeranjima
- mešoviti uslovi

$$\underbrace{\begin{aligned} N_x \cdot \cos\alpha + N_{xy} \cdot \sin\alpha &= p_{nx} \\ N_{xy} \cdot \cos\alpha + N_y \cdot \sin\alpha &= p_{ny} \end{aligned}}_{\text{površinski uslovi}}$$

$$p_{nx} = \frac{\partial^2 F}{\partial y^2} \cdot \frac{dy}{ds} + \frac{\partial^2 F}{\partial x \cdot \partial y} \cdot \frac{dx}{ds} + U \cdot \frac{dy}{ds} = \frac{d}{ds}\left(\frac{\partial F}{\partial y}\right) + U \cdot \frac{dy}{ds}$$

$$p_{nx} = \frac{\partial^2 F}{\partial x^2} \cdot \frac{dx}{ds} + \frac{\partial^2 F}{\partial x \cdot \partial y} \cdot \frac{dy}{ds} + U \cdot \frac{dx}{ds} = \frac{d}{ds}\left(\frac{\partial F}{\partial x}\right) + U \cdot \frac{dx}{ds}$$

$$p^*_{nx} = p_{nx} - U \cdot \frac{dy}{ds} \qquad p^*_{ny} = p_{ny} - U \cdot \frac{dx}{ds}$$

$$\frac{\partial F}{\partial y} = \int_0^s p^*_{nx} \cdot ds + \left(\frac{\partial F}{\partial y}\right)_0 \qquad -\frac{\partial F}{\partial x} = \int_0^s p^*_{ny} \cdot ds + \left(\frac{\partial F}{\partial x}\right)_0$$

$$T_x = \frac{\partial F}{\partial y} = \int_0^s p_{nx}^* \cdot ds \qquad T_y = -\frac{\partial F}{\partial x} = \int_0^s p_{ny}^* \cdot ds$$

$$dF = \frac{\partial F}{\partial x} \cdot dx + \frac{\partial F}{\partial y} \cdot dy$$

$$F = \int_0^s (T_x \cdot dy - T_y \cdot dx)$$

$$F = [T_x \cdot y - T_y \cdot x]\Big|_0^s - \int_0^s (y \cdot dT_x \cdot dy - x \cdot dT_y)$$

$$F = \int_0^s (x - x_s) \cdot p_{ny}^* \cdot ds + \int_0^s (y - y_s) \cdot p_{nx}^* \cdot ds = M$$

Pomeranja tačaka ploče

$$u = \int_0^x \frac{\partial u}{\partial x} \cdot dx + \frac{\partial u}{\partial y} \cdot dy \qquad v = \int_0^x \frac{\partial v}{\partial x} \cdot dx + \frac{\partial v}{\partial y} \cdot dy$$

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$h \cdot E \cdot u = \int (N_x - \nu \cdot N_y) \cdot dx + \Phi(y)$$

$$h \cdot E \cdot v = \int (N_y - \nu \cdot N_x) \cdot dy + \Psi(x)$$

$$h \cdot E \cdot u = \int \frac{\partial^2 F}{\partial y^2} \cdot dx - v \cdot \frac{\partial F}{\partial x} + \Phi(y)$$

$$h \cdot E \cdot u = \int \frac{\partial^2 F}{\partial x^2} \cdot dy - v \cdot \frac{\partial F}{\partial y} + \Psi(x)$$

$$h \cdot E \cdot \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = E \cdot h \cdot \gamma_{xy} = -2 \cdot (1 + v) \cdot N_{xy} = -2 \cdot (1 + v) \cdot \frac{\partial^2 F}{\partial x \cdot \partial y}$$

$$\int \frac{\partial^3 F}{\partial y^3} \cdot dx + \int \frac{\partial^3 F}{\partial x^3} \cdot dy + \frac{d\Phi(y)}{dy} + \frac{d\Psi(x)}{dx} = -2 \cdot \frac{\partial^2 F}{\partial x \cdot \partial y}$$

Ravno stanje deformacije

- tačke poprečnih preseka dugačkih prizmatičnih nosača (brane, nasipi, potporni zidovi...) ostaju u poprečnom preseku i posle dejstva opterećenja - komponentalna pomeranja su $u=u(x,y)$ i $v=v(x,y)$
- svi poprečni preseci imaju isto naponsko-deformacijako stanje bez obzira na njihov položaj, ako su dovoljno udaljeni od osnove-mesta oslanjanja

$$w = 0 \qquad \frac{\partial u}{\partial z} = 0 \qquad \frac{\partial v}{\partial z} = 0$$

$$\Downarrow$$

$$u = u(x, y) \qquad v = v(x, y) \qquad \varepsilon_z = 0 \qquad \gamma_{xz} = 0 \qquad \gamma_{yz} = 0$$

$$E \cdot \varepsilon_z = \sigma_z - (\sigma_x + v \cdot \sigma_y) = 0 \quad \Rightarrow \quad \sigma_z = \sigma_x + v \cdot \sigma_y$$

$$E \cdot \varepsilon_x = \sigma_x - (\sigma_y + v \cdot \sigma_z) = (1 - v^2) \cdot \sigma_x - v \cdot (1 + v) \cdot \sigma_y$$

$$E \cdot \varepsilon_y = (1 - v^2) \cdot \sigma_y - v \cdot (1 + v) \cdot \sigma_x \qquad E \cdot \gamma_{xy} = 2 \cdot (1 + v) \cdot \tau_{xy}$$

$$\frac{\partial \sigma_x}{\partial z} = 0 \qquad \frac{\partial \sigma_y}{\partial z} = 0 \qquad \frac{\partial \sigma_z}{\partial z} = 0 \qquad \frac{\partial \tau_{xy}}{\partial z} = 0$$

$$\Downarrow$$

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0 \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y = 0$$

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \cdot \partial y}$$

$$(1-v) \cdot \left(\frac{\partial^2 \sigma_x}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial x^2} \right) - v \cdot \left(\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2} \right) = 2 \cdot \frac{\partial^2 \tau_{xy}}{\partial x \cdot \partial y}$$

$$\frac{\partial^2\sigma_x}{\partial x^2}+\frac{\partial^2\sigma_y}{\partial y^2}+\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}=-2\cdot\frac{\partial^2\tau_{xy}}{\partial x\cdot\partial y}$$

$$(1-v)\cdot\Delta\cdot(\sigma_x+\sigma_y)+\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}=0$$

$$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

$$X=\frac{\partial U}{\partial x}\qquad Y=\frac{\partial U}{\partial y}\qquad \tau_{xy}=-\frac{\partial^2 F}{\partial x\cdot\partial y}$$

$$\frac{\partial}{\partial x}\left(\sigma_x - \frac{\partial^2 F}{\partial y^2} - U\right) = 0 \Rightarrow \sigma_x = \frac{\partial^2 F}{\partial y^2} + U$$

$$\frac{\partial}{\partial x}\left(\sigma_x - \frac{\partial^2 F}{\partial y^2} - U\right) = 0 \Rightarrow \sigma_x - \frac{\partial^2 F}{\partial y^2} - U$$

$$\Delta\Delta F + \frac{1 - 2 \cdot v}{1 - v} \cdot \Delta U = 0$$

$$\Delta\Delta F = 0$$

Ravan problem u polarnim koordinatama

$$\Delta\Delta F = \left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{r \cdot \partial r} + \frac{\partial^2}{r^2 \cdot \partial \varphi^2}\right)\left(\frac{\partial^2 F}{\partial r^2} + \frac{\partial F}{r \cdot \partial r} + \frac{\partial^2 F}{r^2 \cdot \partial \varphi^2}\right) = 0$$

$$N_r = \frac{\partial^2 F}{r^2 \cdot \partial \varphi^2} + \frac{\partial F}{r \cdot \partial r}$$

$$N_\varphi = \frac{\partial^2 F}{\partial r^2}$$

$$N_{r\varphi} = -\frac{\partial}{\partial r}\left(\frac{\partial F}{r \cdot \partial \varphi}\right)$$

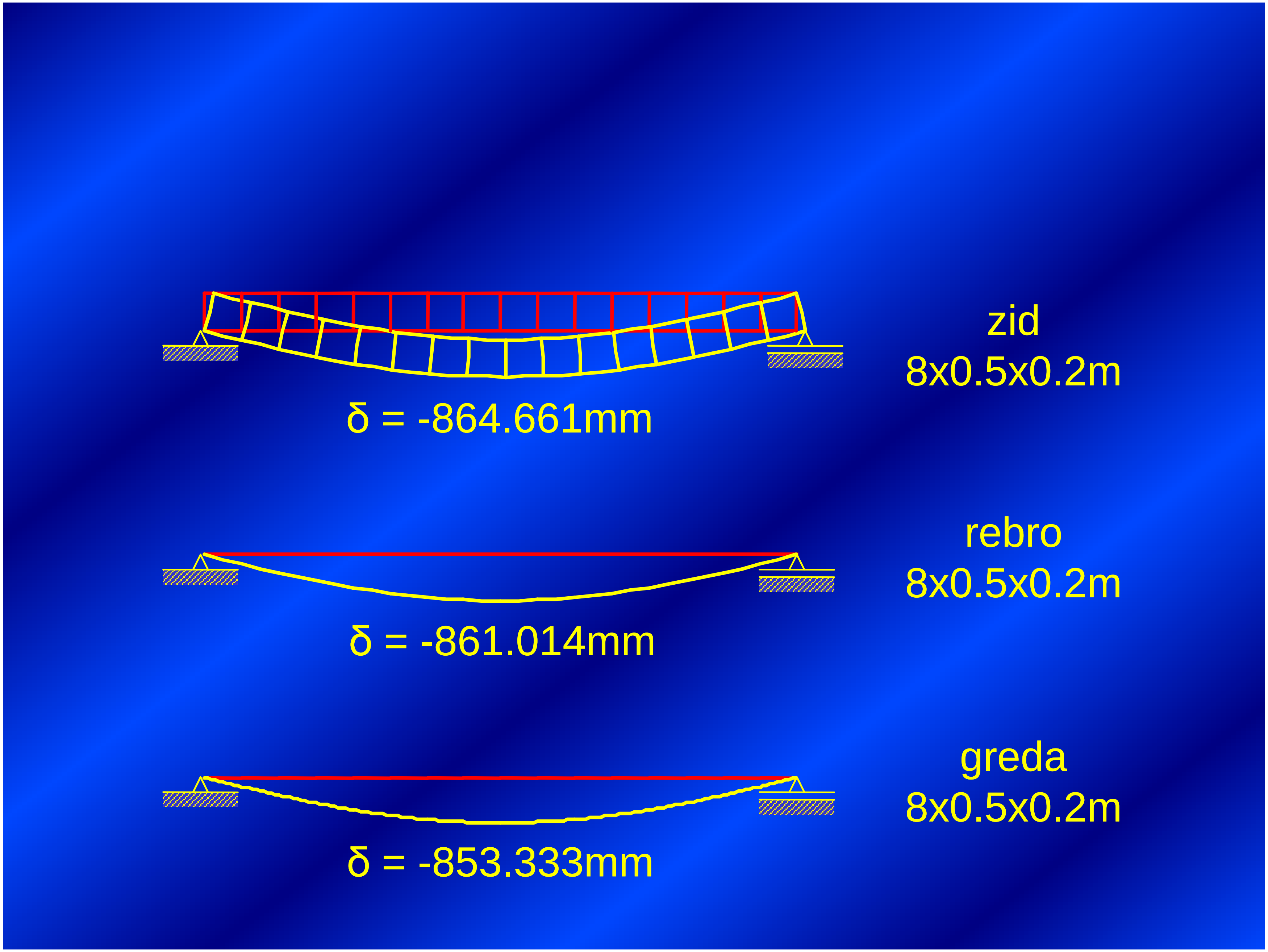
Ravan problem u polarnim koordinatama - rotaciona simetrija

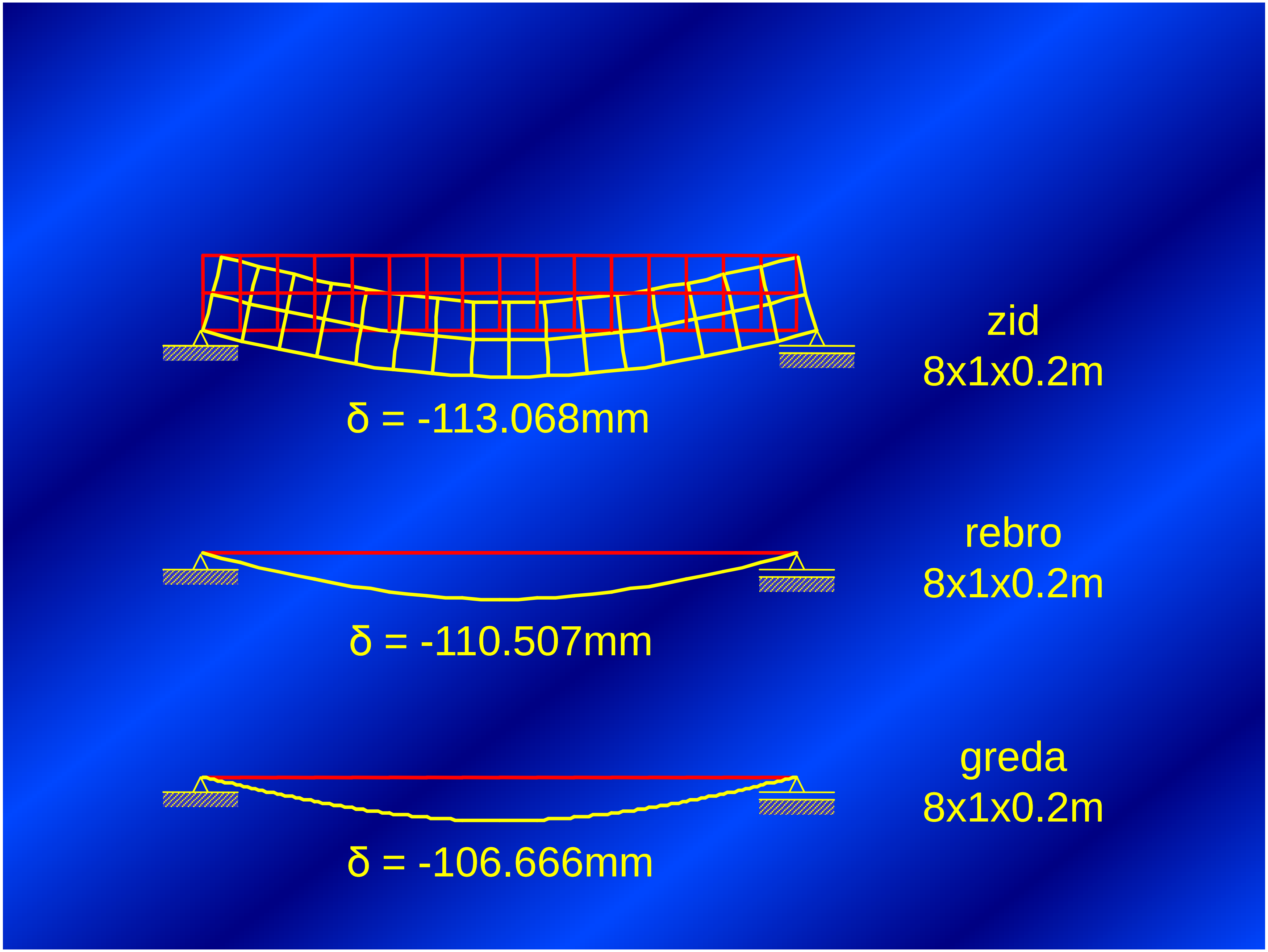
$$\Delta\Delta F = \left(\frac{d^2}{dr^2} + \frac{d}{r \cdot dr}\right)\left(\frac{d^2F}{dr^2} + \frac{dF}{r \cdot dr}\right) = 0$$

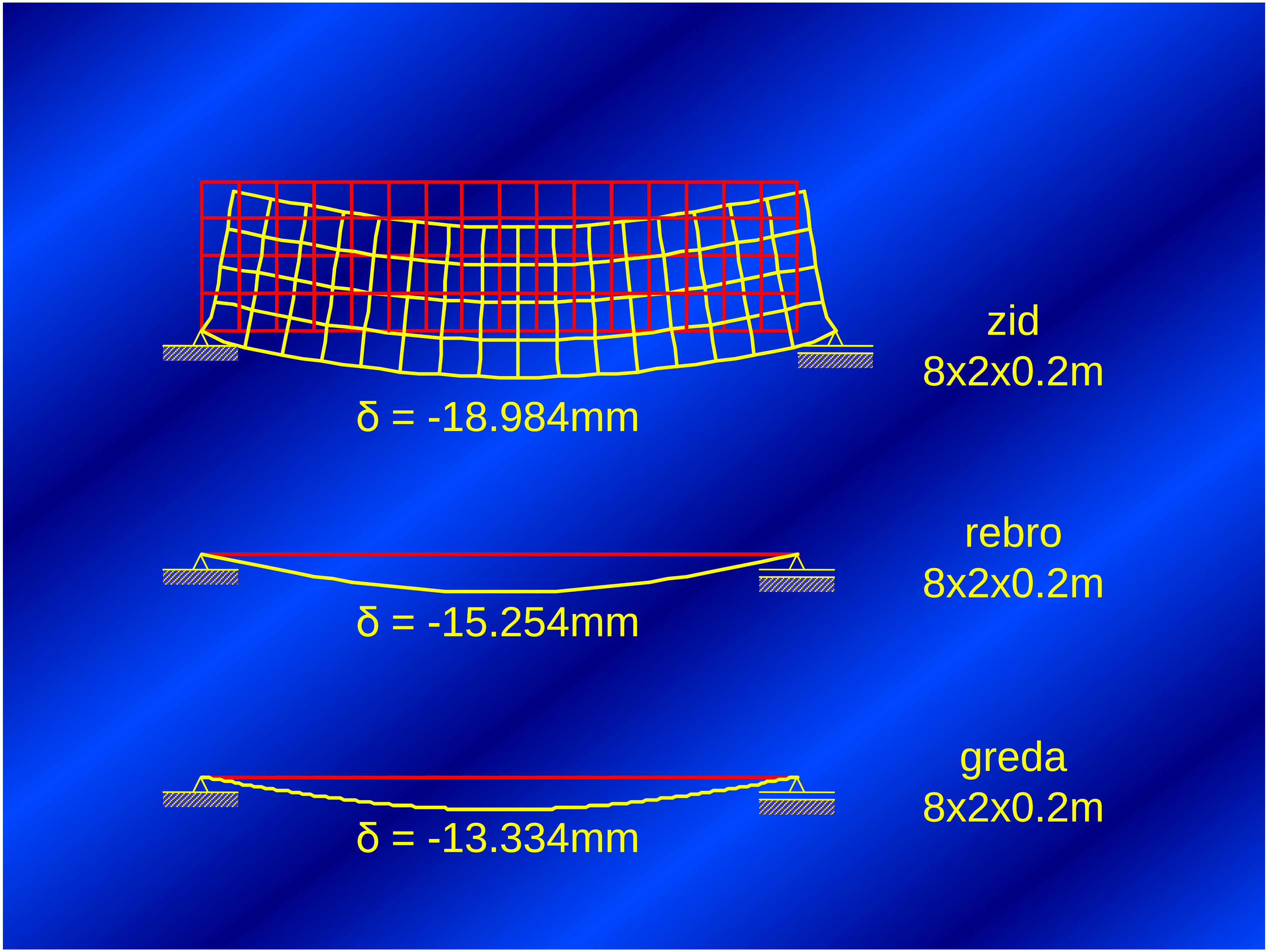
$$N_r = \frac{dF}{r \cdot dr} \qquad N_\varphi = \frac{d^2F}{dr^2} \qquad N_{r\varphi} = 0$$

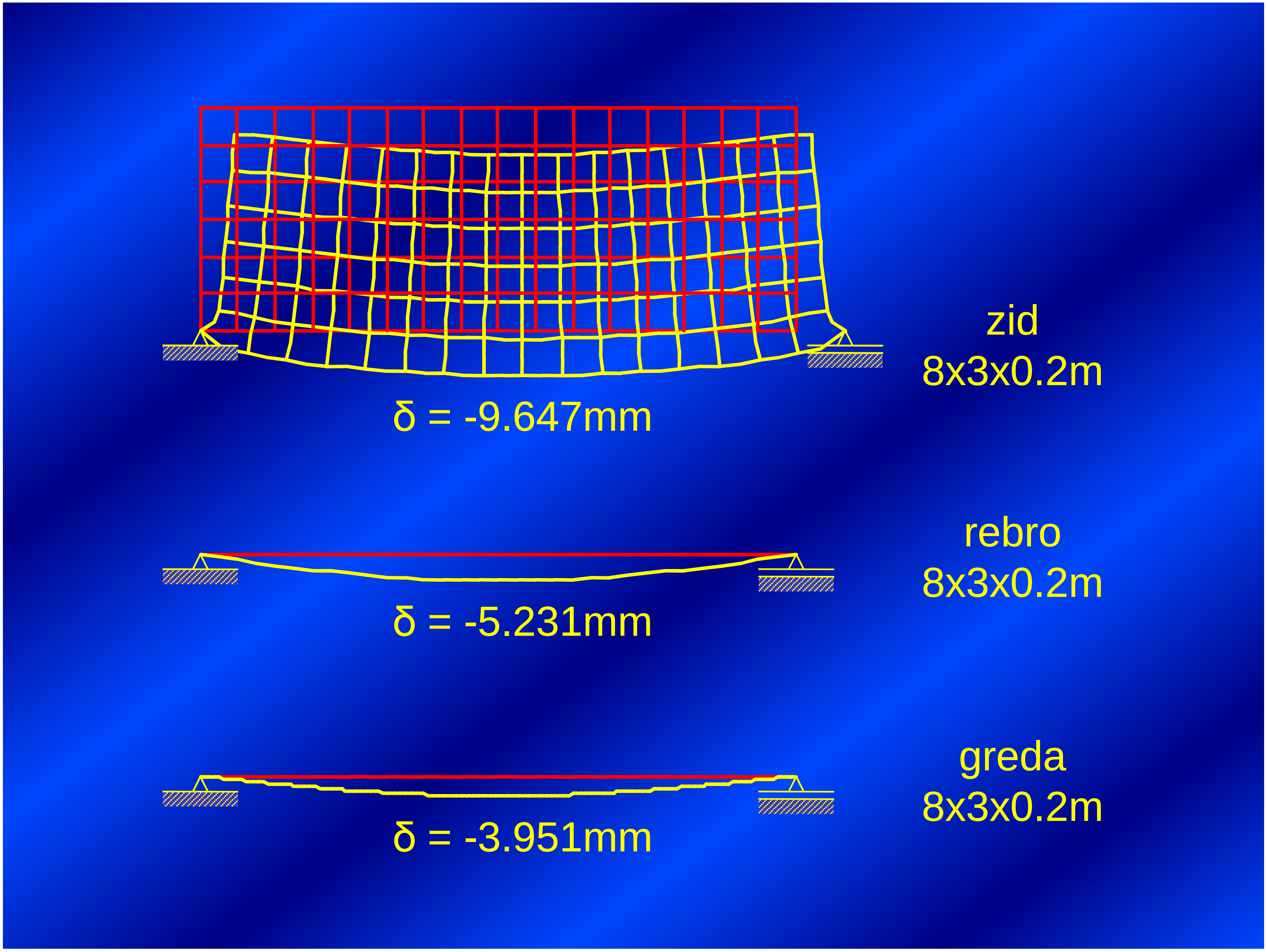
$$\frac{d^4F}{dr^4} + \frac{d^3F}{r \cdot dr^3} + \frac{d^2F}{r^2 \cdot dr^2} + \frac{dF}{r^3 \cdot dr} = 0 \qquad F = D + A \cdot \ln r + B \cdot r^2 + C \cdot r^2 \cdot \ln r$$

$$N_r = \frac{A}{r^2} + 2 \cdot B + C \cdot (1 + 2\ln r) \qquad N_\varphi = -\frac{A}{r^2} + 2 \cdot B + C \cdot (3 + 2\ln r)$$

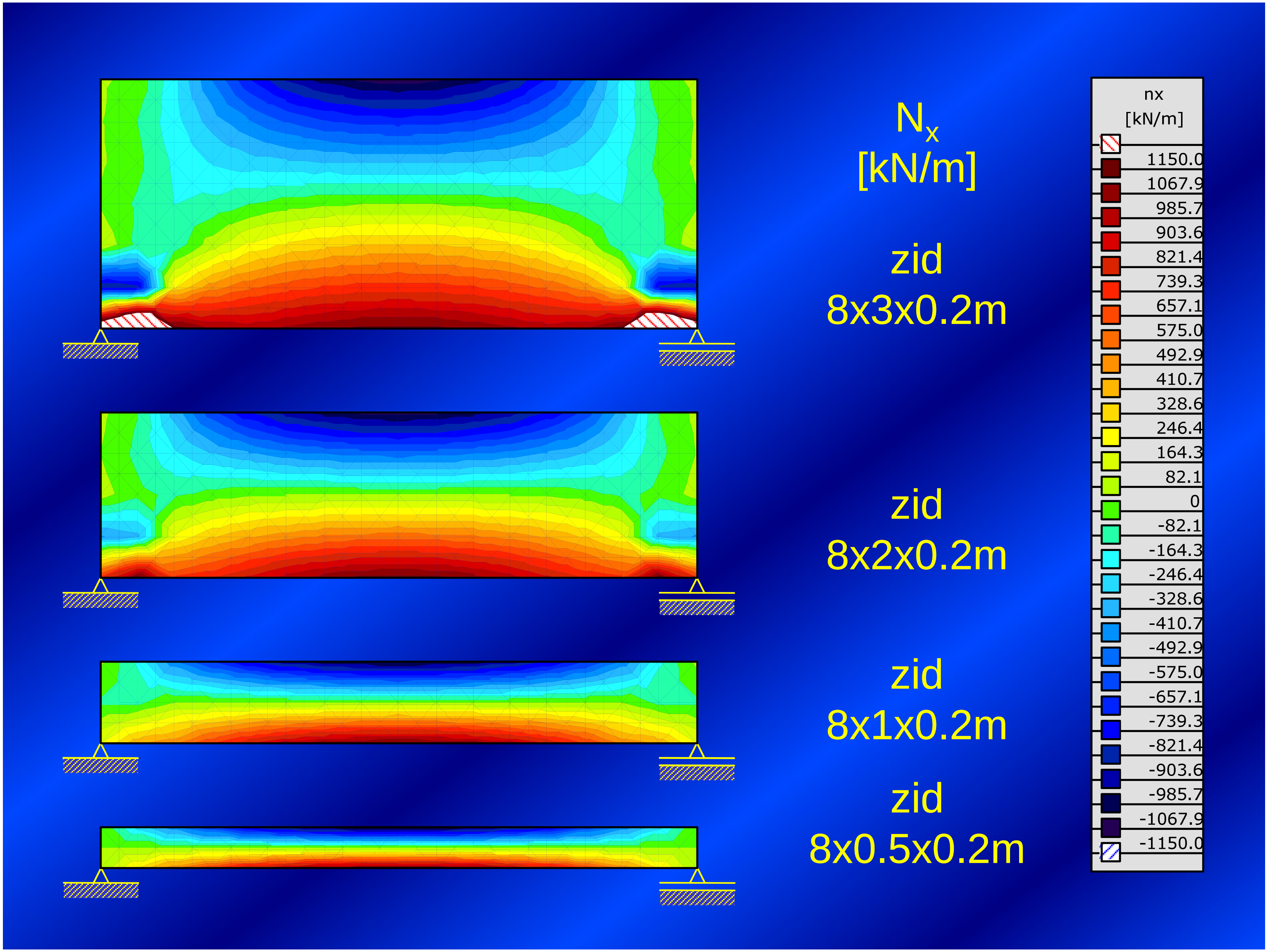

zid
8x0.5x0.2m
δ = -864.661mm
rebro
8x0.5x0.2m
δ = -861.014mm
greda
8x0.5x0.2m
δ = -853.333mm


zid
8x1x0.2m
δ = -113.068mm
rebro
8x1x0.2m
δ = -110.507mm
greda
8x1x0.2m
δ = -106.666mm

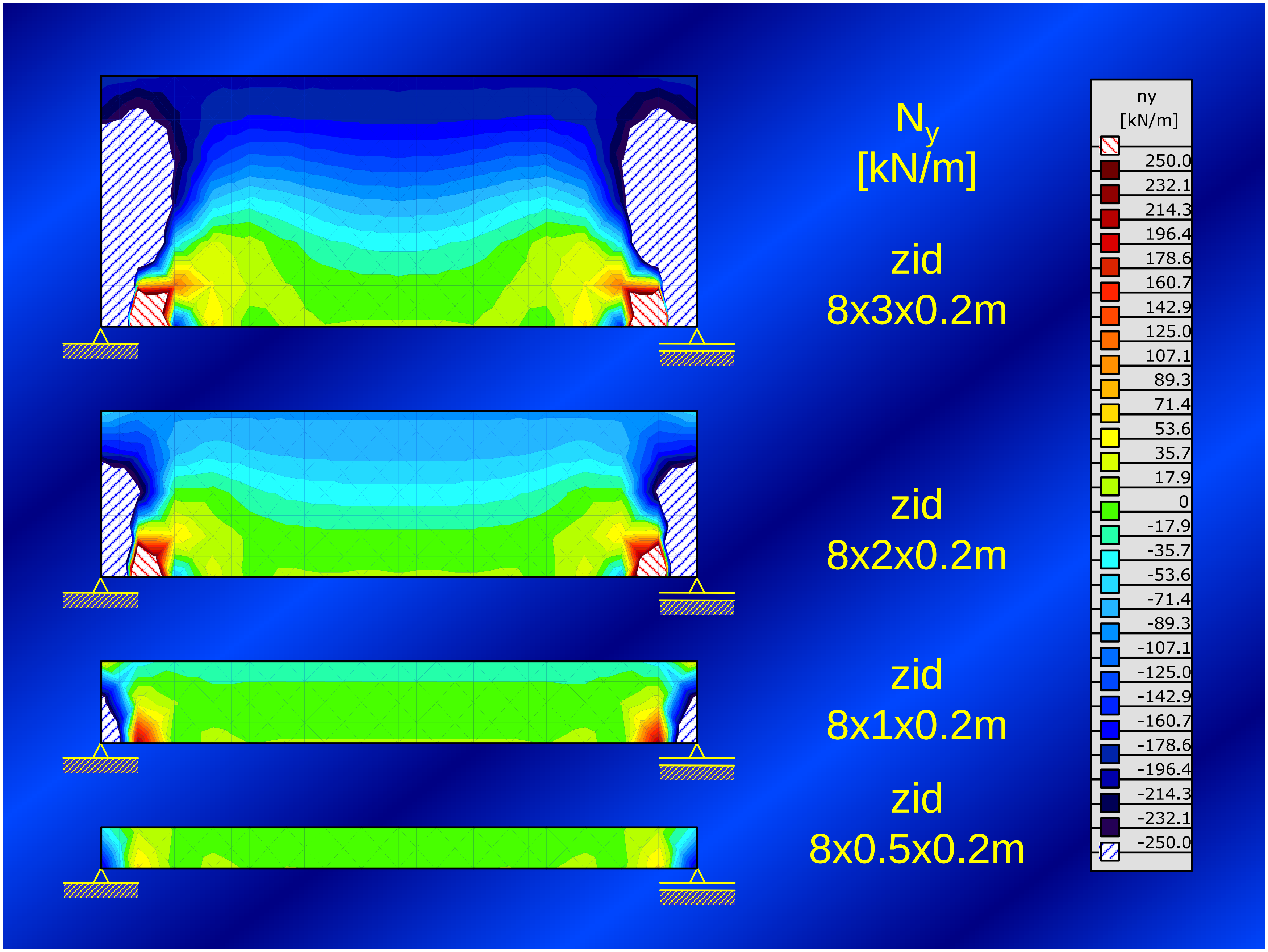

zid
8x2x0.2m
δ = -18.984mm
rebro
8x2x0.2m
δ = -15.254mm
greda
8x2x0.2m
δ = -13.334mm



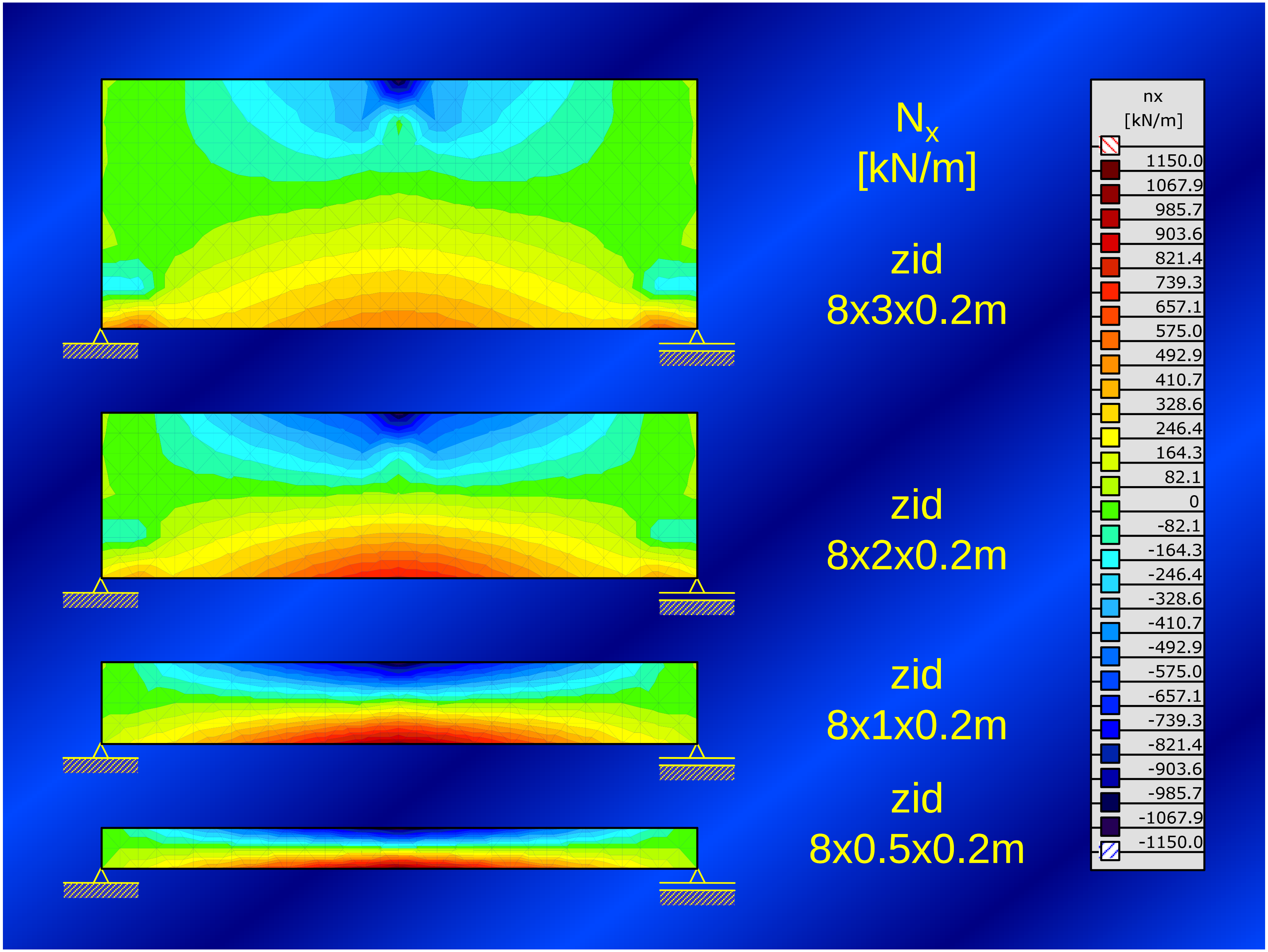
zid
8x3x0.2m
δ = -9.647mm
rebro
8x3x0.2m
δ = -5.231mm
greda
8x3x0.2m
δ = -3.951mm



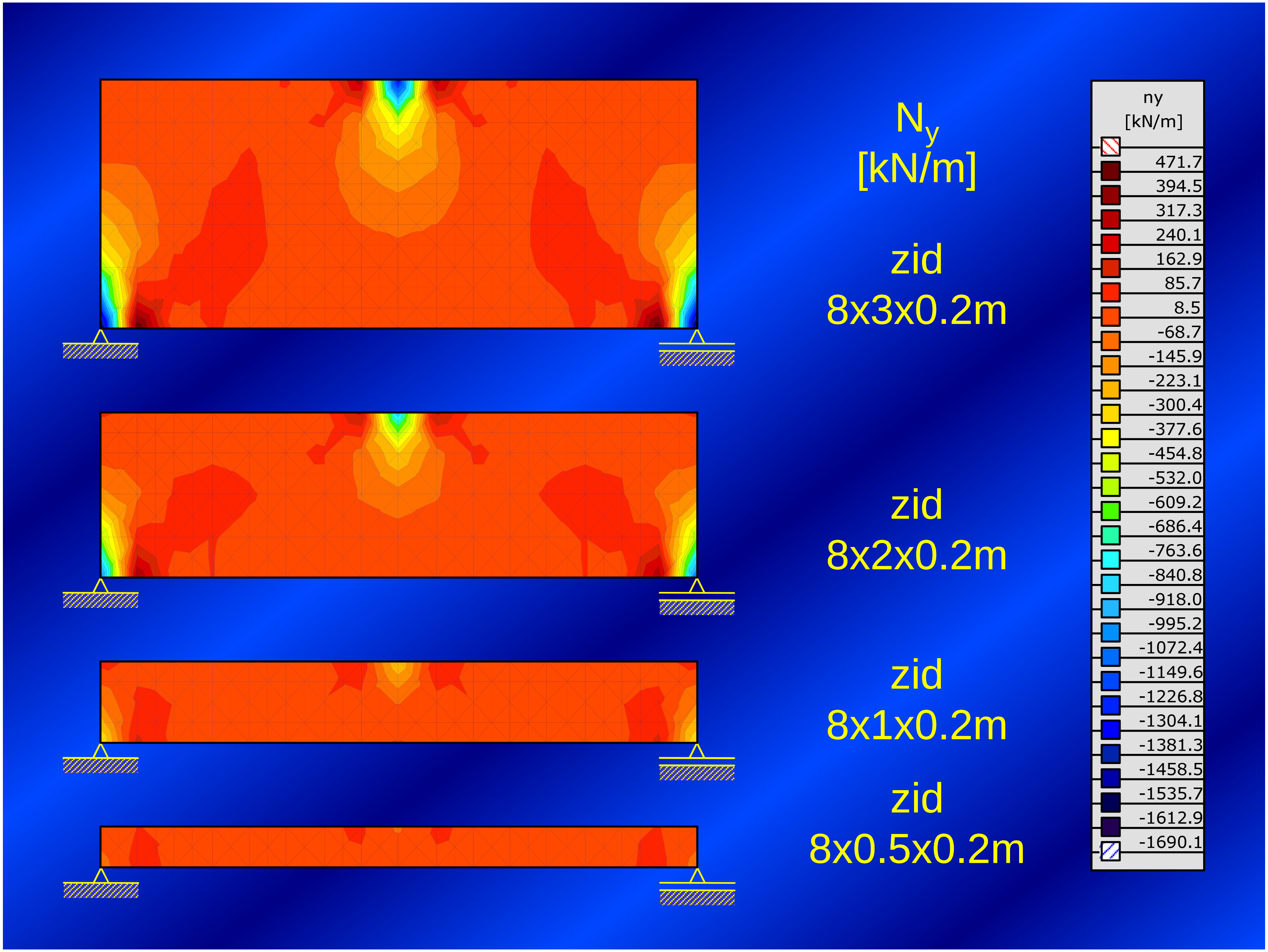
N_x [kN/m]
zid 8x3x0.2m
zid 8x2x0.2m
zid 8x1x0.2m
zid 8x0.5x0.2m
nx [kN/m]
1150.0
1067.9
985.7
903.6
821.4
739.3
657.1
575.0
492.9
410.7
328.6
246.4
164.3
82.1
0
-82.1
-164.3
-246.4
-328.6
-410.7
-492.9
-575.0
-657.1
-739.3
-821.4
-903.6
-985.7
-1067.9
-1150.0

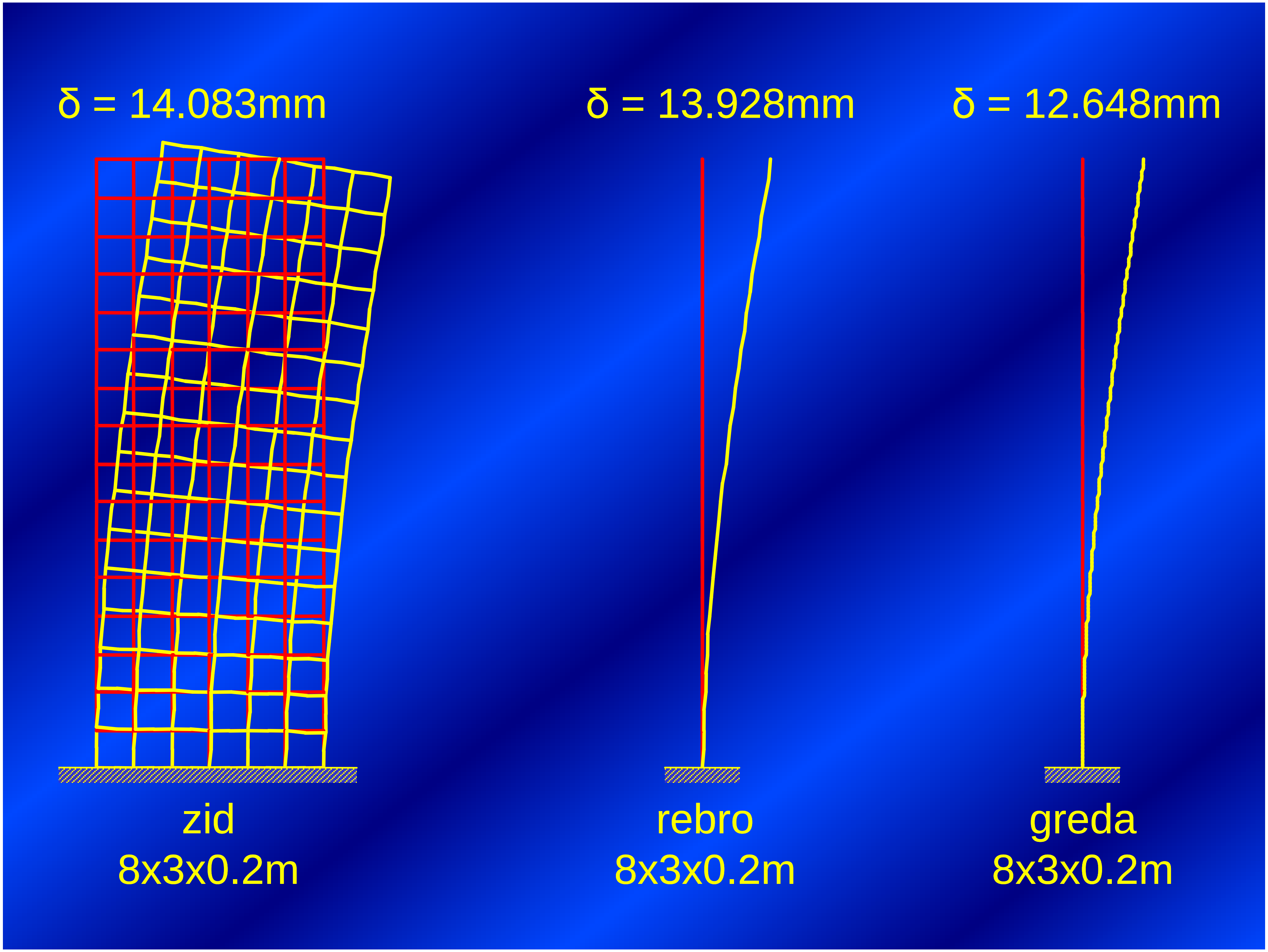


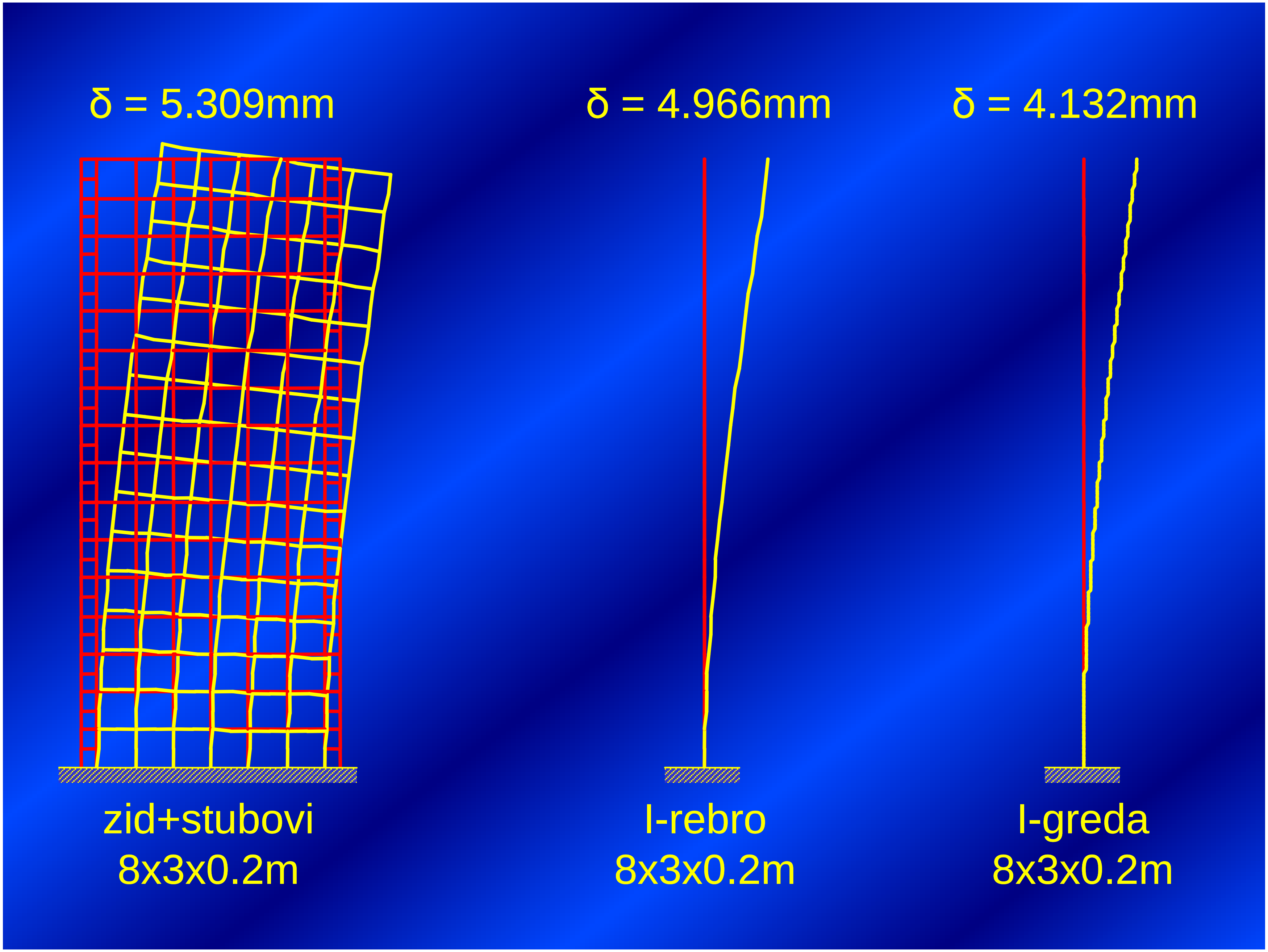
N_y
[kN/m]
zid
8x3x0.2m
zid
8x2x0.2m
zid
8x1x0.2m
zid
8x0.5x0.2m
ny
[kN/m]
250.0
232.1
214.3
196.4
178.6
160.7
142.9
125.0
107.1
89.3
71.4
53.6
35.7
17.9
0
-17.9
-35.7
-53.6
-71.4
-89.3
-107.1
-125.0
-142.9
-160.7
-178.6
-196.4
-214.3
-232.1
-250.0

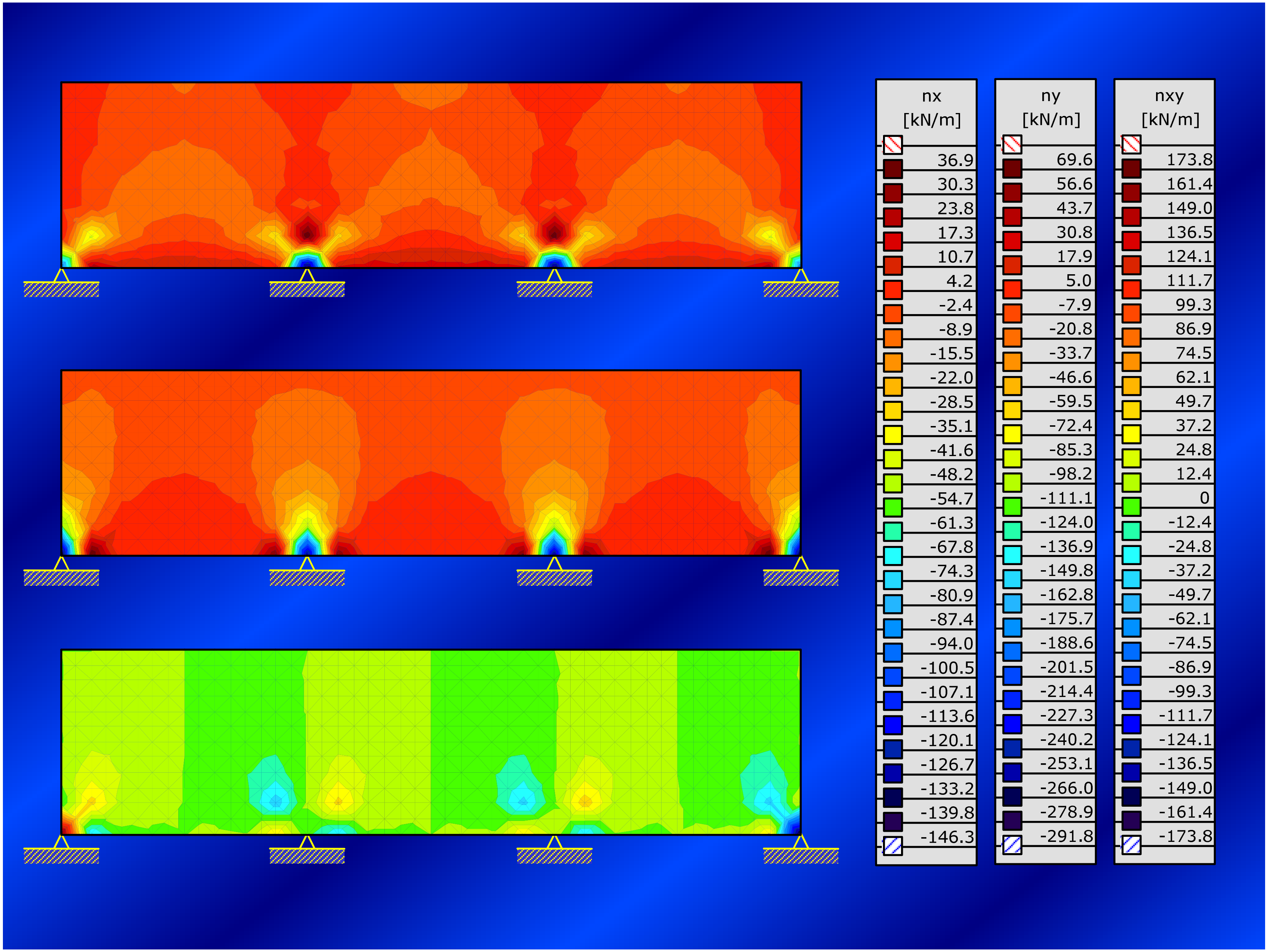


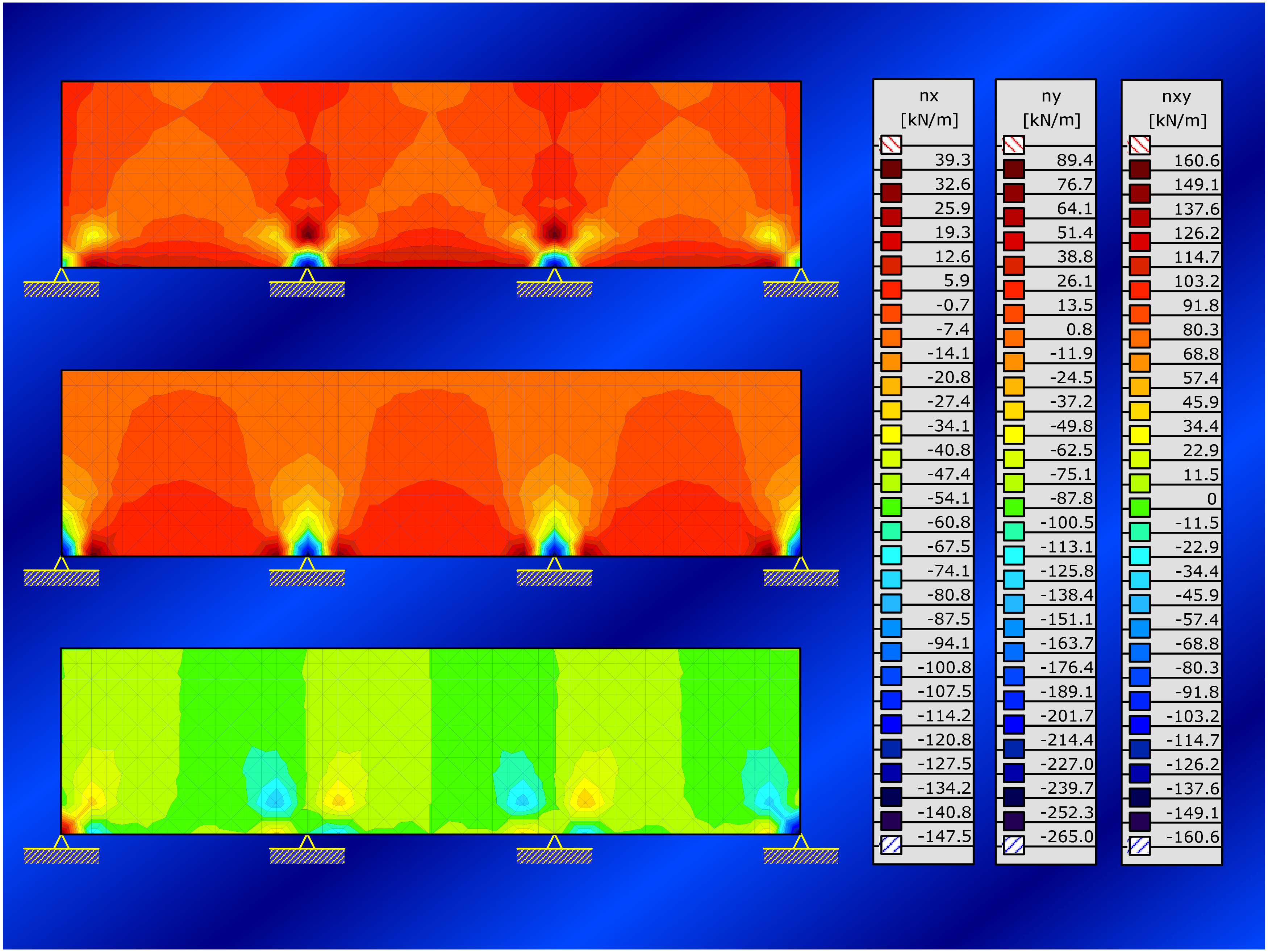
N_x [kN/m]
zid 8x3x0.2m
zid 8x2x0.2m
zid 8x1x0.2m
zid 8x0.5x0.2m
nx [kN/m]
1150.0
1067.9
985.7
903.6
821.4
739.3
657.1
575.0
492.9
410.7
328.6
246.4
164.3
82.1
0
-82.1
-164.3
-246.4
-328.6
-410.7
-492.9
-575.0
-657.1
-739.3
-821.4
-903.6
-985.7
-1067.9
-1150.0


N_y
[kN/m]
zid
8x3x0.2m
zid
8x2x0.2m
zid
8x1x0.2m
zid
8x0.5x0.2m
ny
[kN/m]
471.7
394.5
317.3
240.1
162.9
85.7
8.5
-68.7
-145.9
-223.1
-300.4
-377.6
-454.8
-532.0
-609.2
-686.4
-763.6
-840.8
-918.0
-995.2
-1072.4
-1149.6
-1226.8
-1304.1
-1381.3
-1458.5
-1535.7
-1612.9
-1690.1

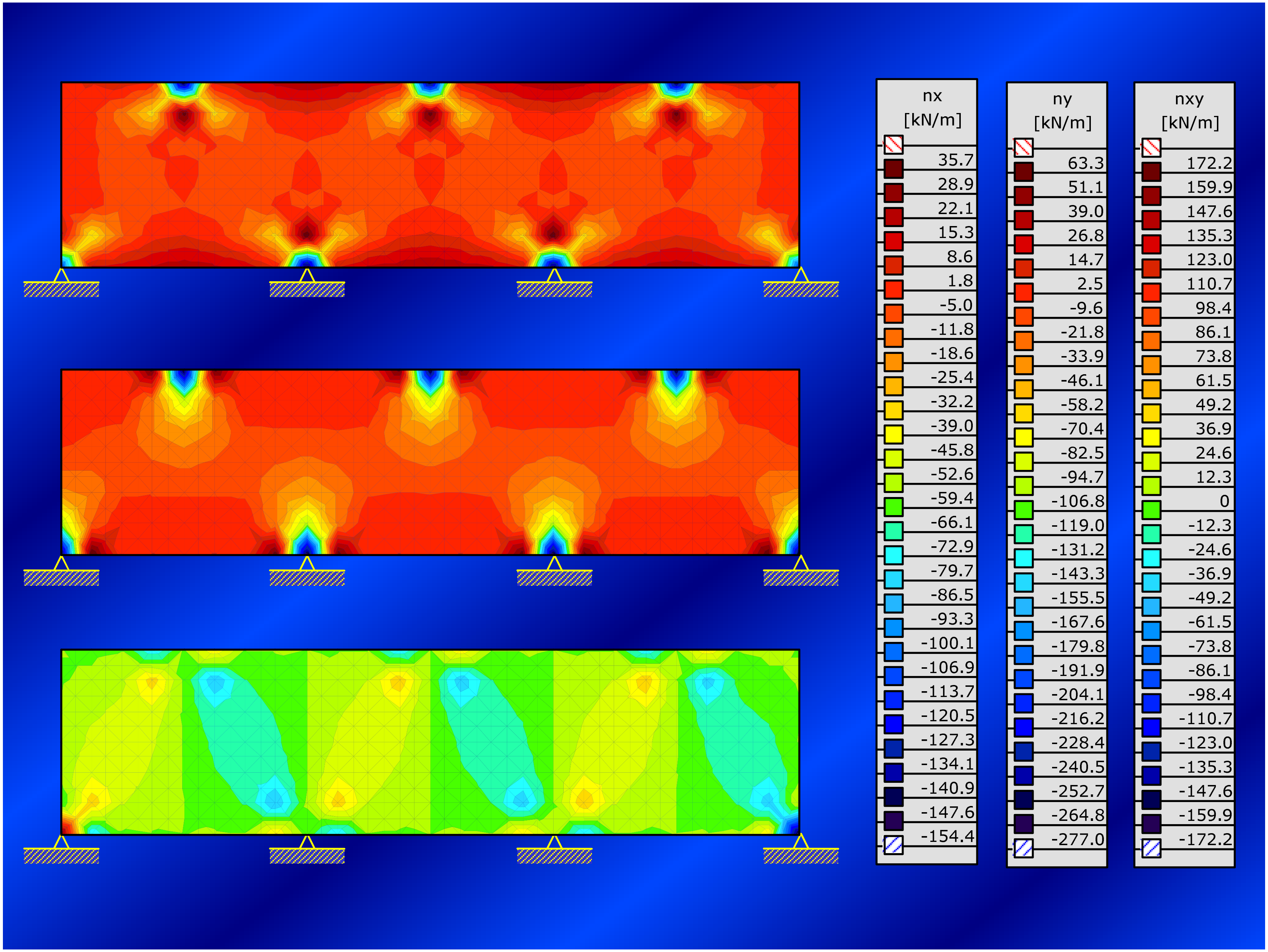

δ = 14.083mm
δ = 13.928mm
δ = 12.648mm
zid
8x3x0.2m
rebro
8x3x0.2m
greda
8x3x0.2m


δ = 5.309mm
δ = 4.966mm
δ = 4.132mm
zid+stubovi
8x3x0.2m
I-rebro
8x3x0.2m
I-greda
8x3x0.2m

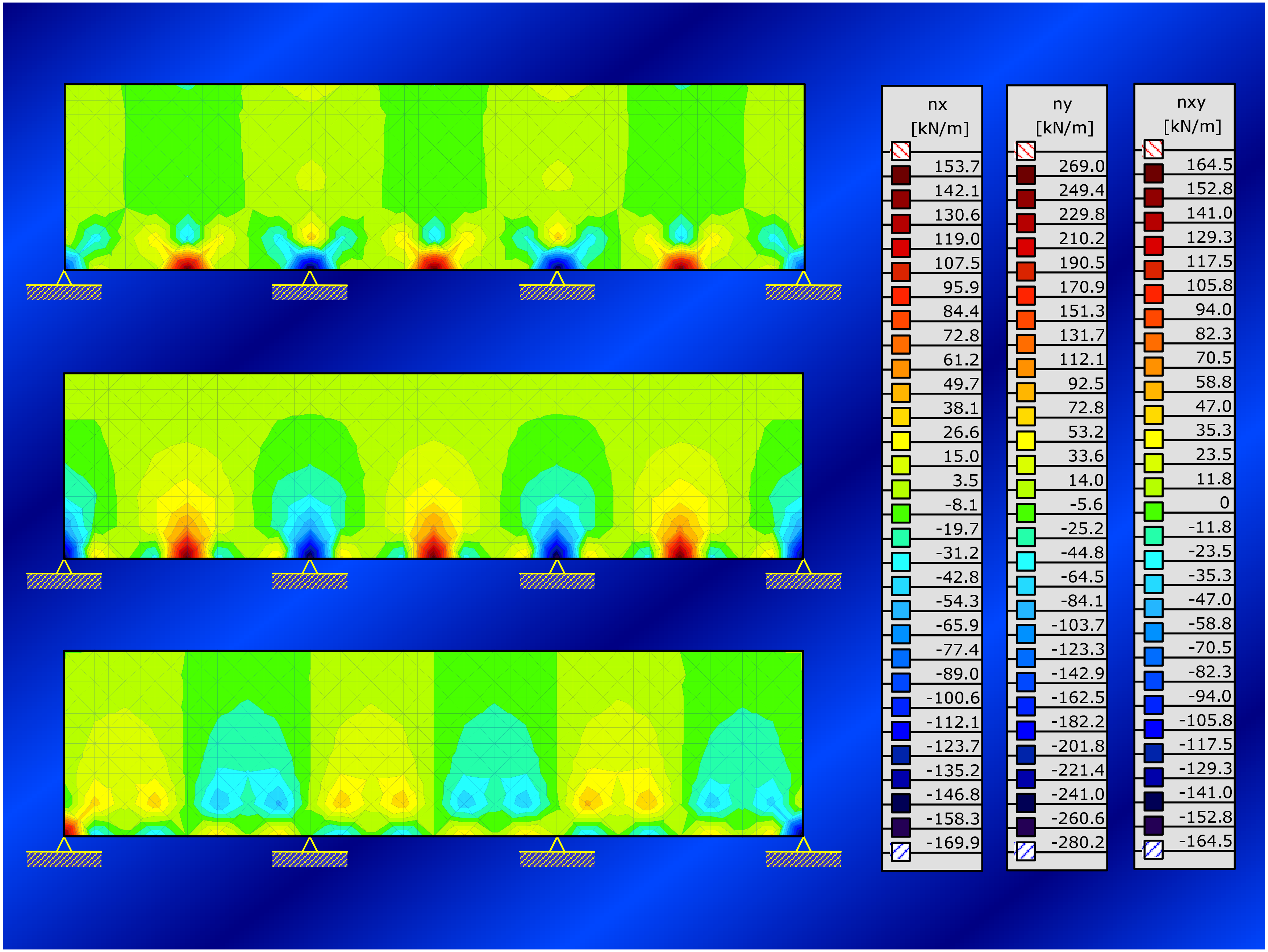

nx [kN/m]
36.9 30.3 23.8 17.3 10.7 4.2 -2.4 -8.9 -15.5 -22.0 -28.5 -35.1 -41.6 -48.2 -54.7 -61.3 -67.8 -74.3 -80.9 -87.4 -94.0 -100.5 -107.1 -113.6 -120.1 -126.7 -133.2 -139.8 -146.3
ny [kN/m]
69.6 56.6 43.7 30.8 17.9 5.0 -7.9 -20.8 -33.7 -46.6 -59.5 -72.4 -85.3 -98.2 -111.1 -124.0 -136.9 -149.8 -162.8 -175.7 -188.6 -201.5 -214.4 -227.3 -240.2 -253.1 -266.0 -278.9 -291.8
nxy [kN/m]
173.8 161.4 149.0 136.5 124.1 111.7 99.3 86.9 74.5 62.1 49.7 37.2 24.8 12.4 0 -12.4 -24.8 -37.2 -49.7 -62.1 -74.5 -86.9 -99.3 -111.7 -124.1 -136.5 -149.0 -161.4 -173.8



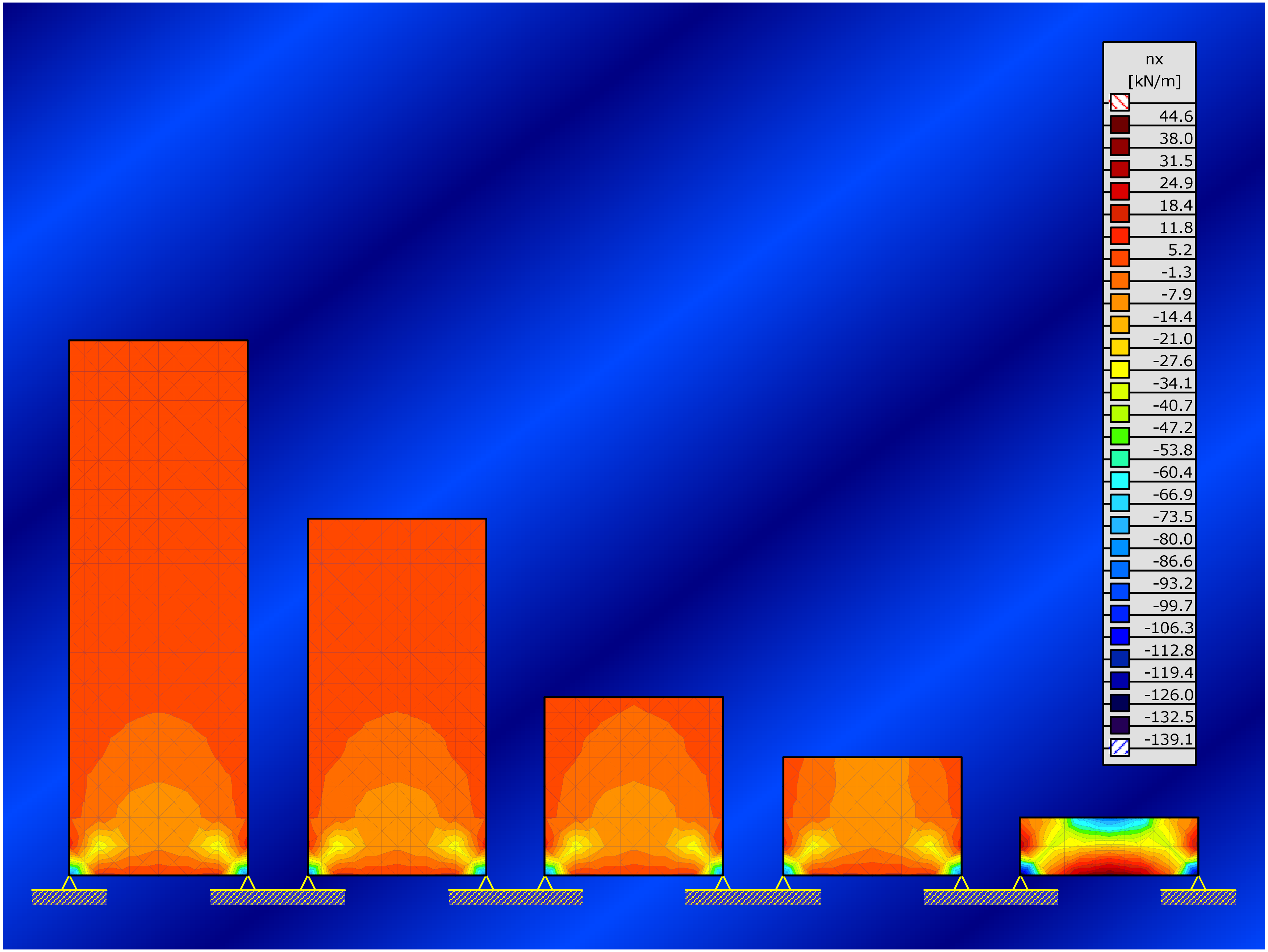
nx
[kN/m]
39.3
32.6
25.9
19.3
12.6
5.9
-0.7
-7.4
-14.1
-20.8
-27.4
-34.1
-40.8
-47.4
-54.1
-60.8
-67.5
-74.1
-80.8
-87.5
-94.1
-100.8
-107.5
-114.2
-120.8
-127.5
-134.2
-140.8
-147.5
ny
[kN/m]
89.4
76.7
64.1
51.4
38.8
26.1
13.5
0.8
-11.9
-24.5
-37.2
-49.8
-62.5
-75.1
-87.8
-100.5
-113.1
-125.8
-138.4
-151.1
-163.7
-176.4
-189.1
-201.7
-214.4
-227.0
-239.7
-252.3
-265.0
nxy
[kN/m]
160.6
149.1
137.6
126.2
114.7
103.2
91.8
80.3
68.8
57.4
45.9
34.4
22.9
11.5
0
-11.5
-22.9
-34.4
-45.9
-57.4
-68.8
-80.3
-91.8
-103.2
-114.7
-126.2
-137.6
-149.1
-160.6

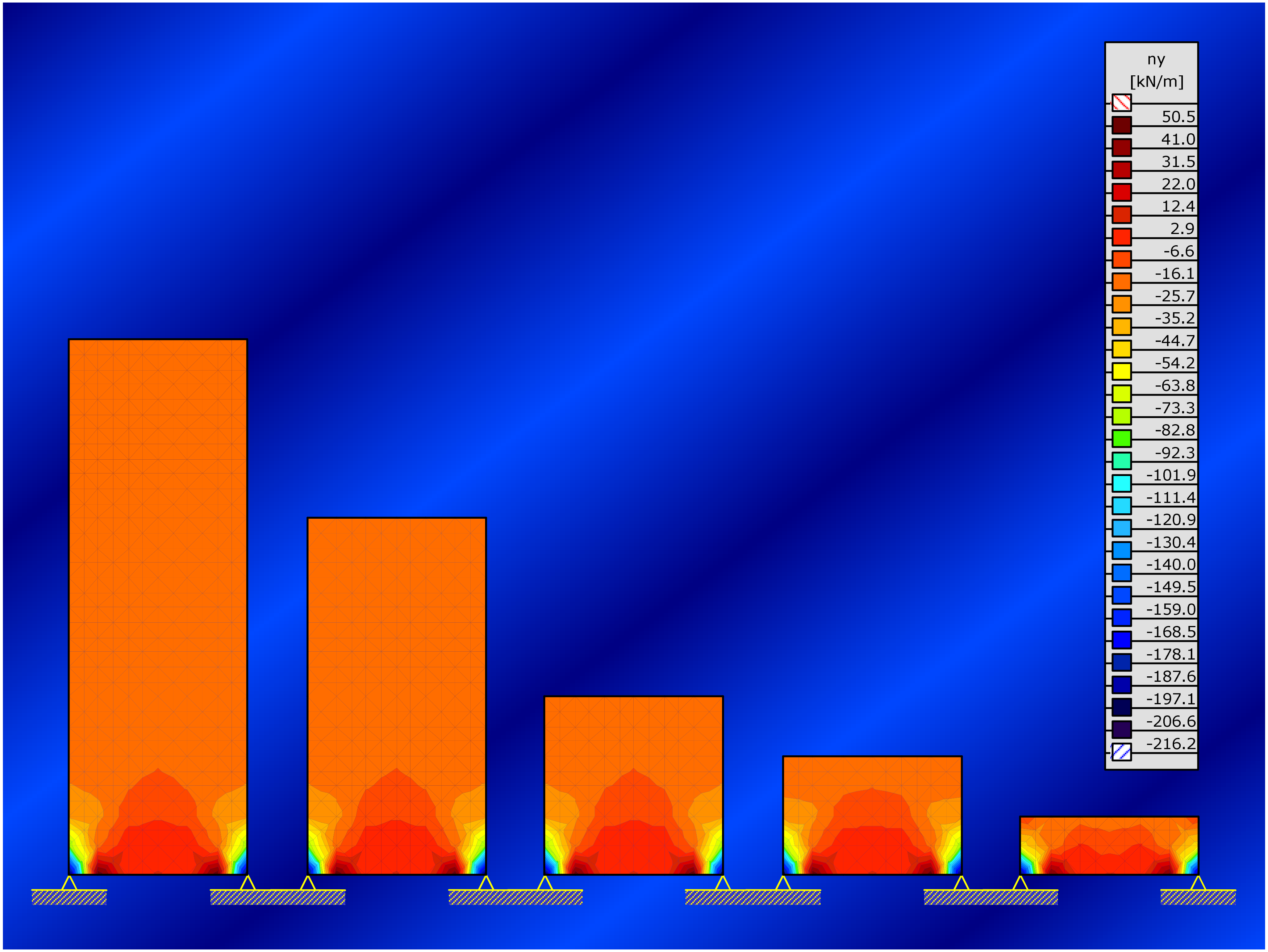


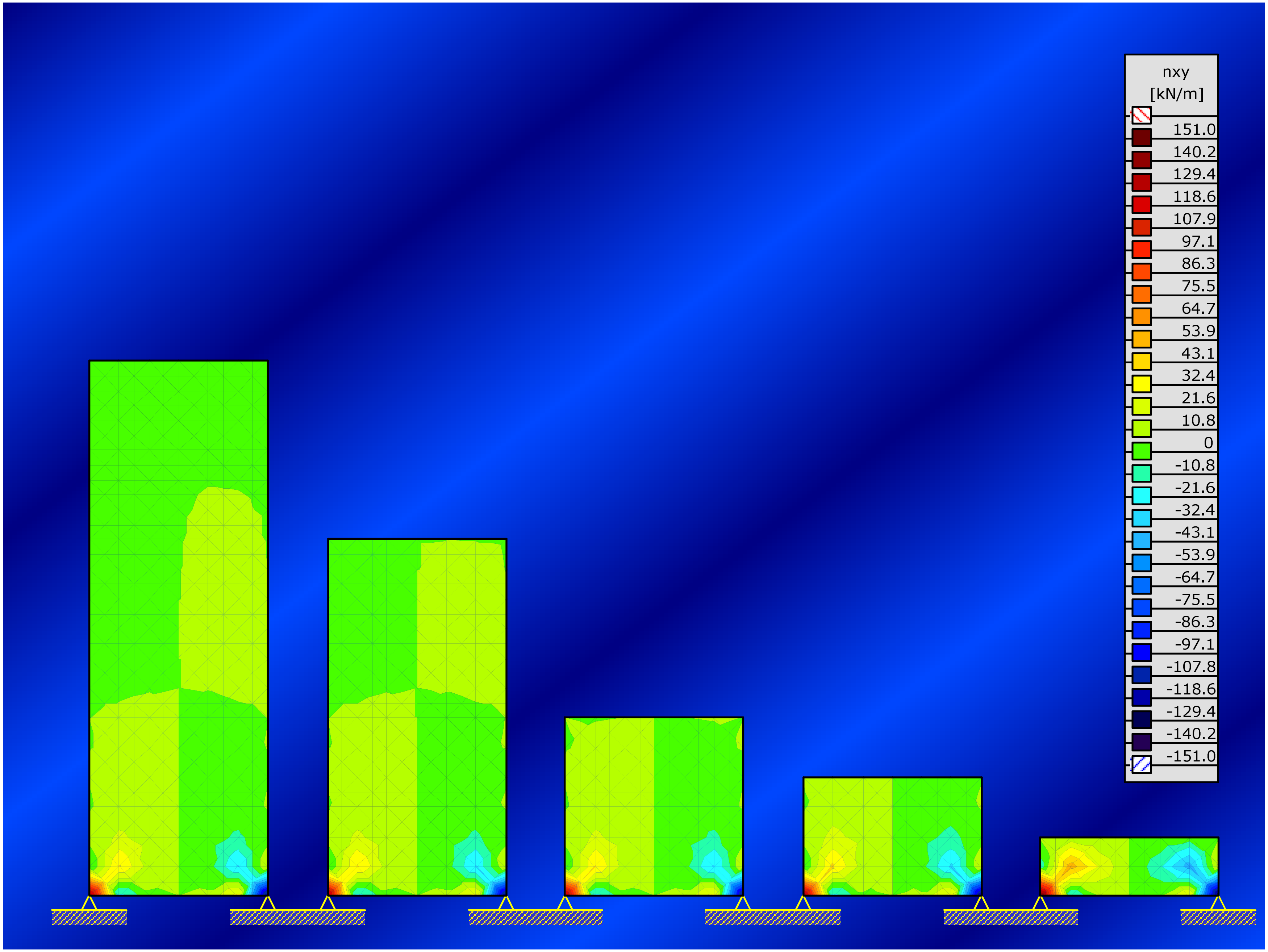
nx
[kN/m]
35.7
28.9
22.1
15.3
8.6
1.8
-5.0
-11.8
-18.6
-25.4
-32.2
-39.0
-45.8
-52.6
-59.4
-66.1
-72.9
-79.7
-86.5
-93.3
-100.1
-106.9
-113.7
-120.5
-127.3
-134.1
-140.9
-147.6
-154.4
ny
[kN/m]
63.3
51.1
39.0
26.8
14.7
2.5
-9.6
-21.8
-33.9
-46.1
-58.2
-70.4
-82.5
-94.7
-106.8
-119.0
-131.2
-143.3
-155.5
-167.6
-179.8
-191.9
-204.1
-216.2
-228.4
-240.5
-252.7
-264.8
-277.0
nxy
[kN/m]
172.2
159.9
147.6
135.3
123.0
110.7
98.4
86.1
73.8
61.5
49.2
36.9
24.6
12.3
0
-12.3
-24.6
-36.9
-49.2
-61.5
-73.8
-86.1
-98.4
-110.7
-123.0
-135.3
-147.6
-159.9
-172.2

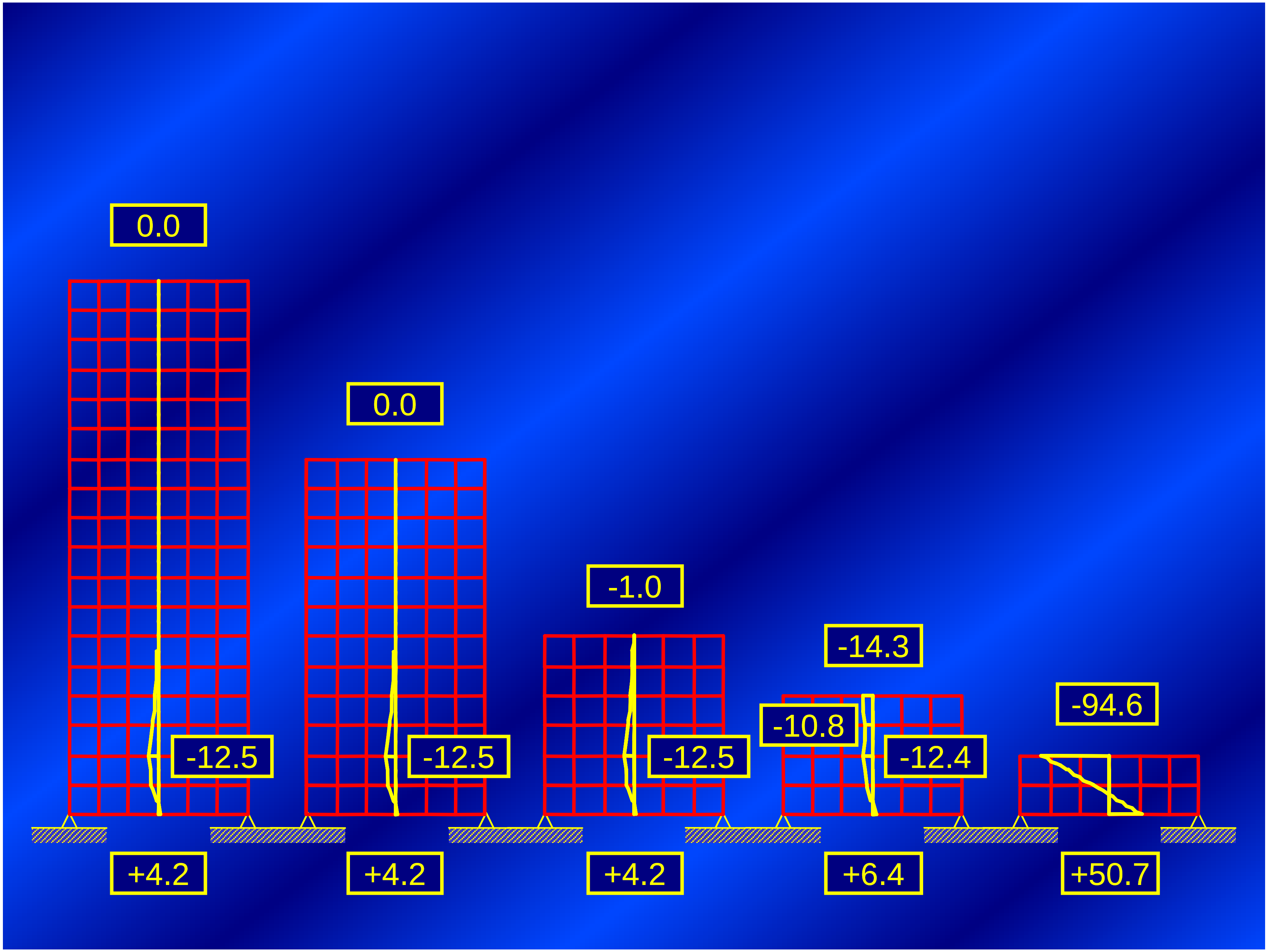


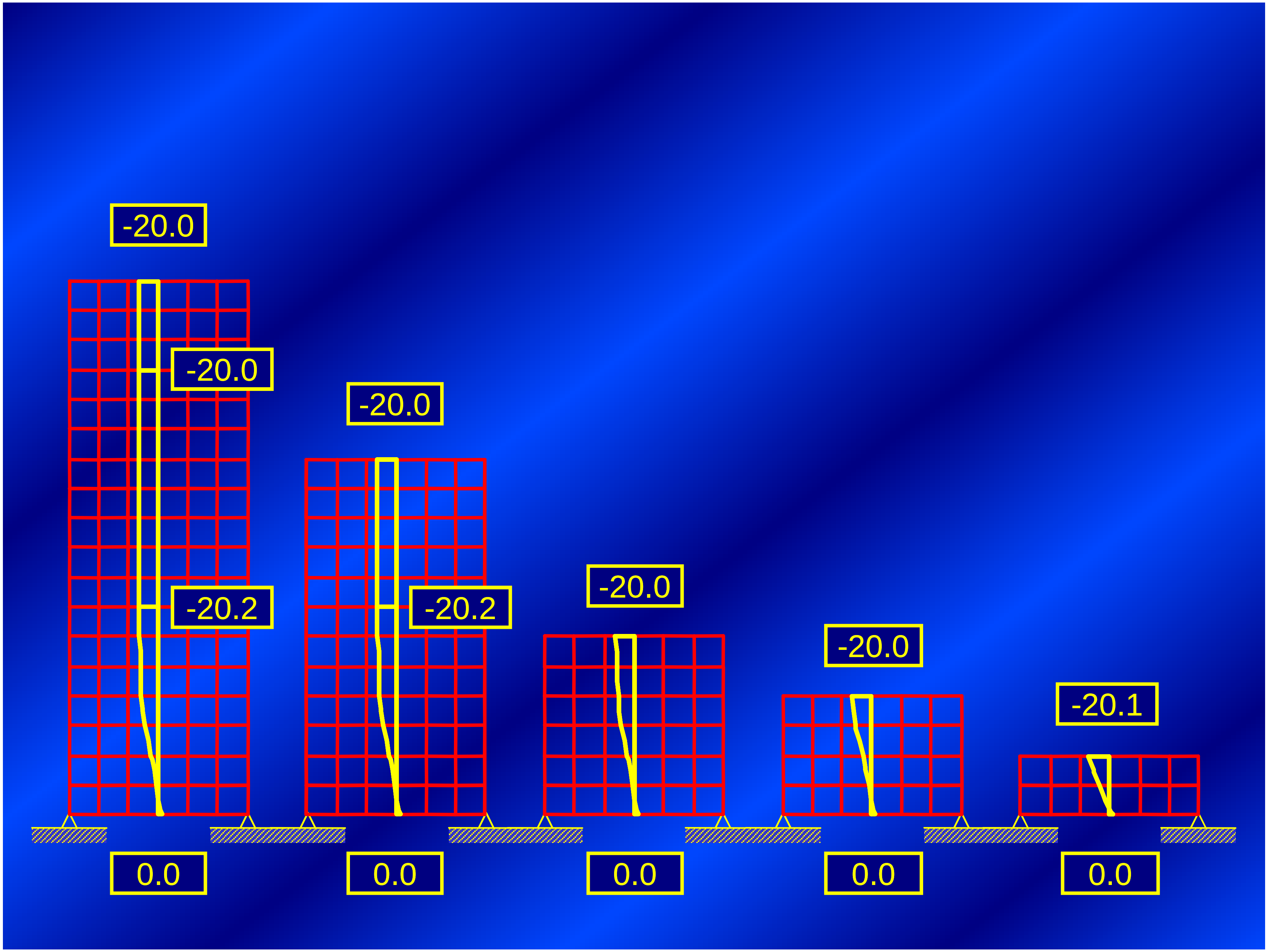
nx
[kN/m]
153.7
142.1
130.6
119.0
107.5
95.9
84.4
72.8
61.2
49.7
38.1
26.6
15.0
3.5
-8.1
-19.7
-31.2
-42.8
-54.3
-65.9
-77.4
-89.0
-100.6
-112.1
-123.7
-135.2
-146.8
-158.3
-169.9
ny
[kN/m]
269.0
249.4
229.8
210.2
190.5
170.9
151.3
131.7
112.1
92.5
72.8
53.2
33.6
14.0
-5.6
-25.2
-44.8
-64.5
-84.1
-103.7
-123.3
-142.9
-162.5
-182.2
-201.8
-221.4
-241.0
-260.6
-280.2
nxy
[kN/m]
164.5
152.8
141.0
129.3
117.5
105.8
94.0
82.3
70.5
58.8
47.0
35.3
23.5
11.8
0
-11.8
-23.5
-35.3
-47.0
-58.8
-70.5
-82.3
-94.0
-105.8
-117.5
-129.3
-141.0
-152.8
-164.5


nx
[kN/m]
44.6
38.0
31.5
24.9
18.4
11.8
5.2
-1.3
-7.9
-14.4
-21.0
-27.6
-34.1
-40.7
-47.2
-53.8
-60.4
-66.9
-73.5
-80.0
-86.6
-93.2
-99.7
-106.3
-112.8
-119.4
-126.0
-132.5
-139.1


ny
[kN/m]
50.5
41.0
31.5
22.0
12.4
2.9
-6.6
-16.1
-25.7
-35.2
-44.7
-54.2
-63.8
-73.3
-82.8
-92.3
-101.9
-111.4
-120.9
-130.4
-140.0
-149.5
-159.0
-168.5
-178.1
-187.6
-197.1
-206.6
-216.2


nxy
[kN/m]
151.0
140.2
129.4
118.6
107.9
97.1
86.3
75.5
64.7
53.9
43.1
32.4
21.6
10.8
0
-10.8
-21.6
-32.4
-43.1
-53.9
-64.7
-75.5
-86.3
-97.1
-107.8
-118.6
-129.4
-140.2
-151.0


0.0
0.0
-1.0
-14.3
-94.6
-10.8
-12.5
-12.5
-12.5
-12.4
+4.2
+4.2
+4.2
+6.4
+50.7


-20.0
-20.0
-20.2
0.0
-20.0
-20.2
0.0
-20.0
0.0
-20.0
0.0
-20.1
0.0

LJUSKE

Osnovni pojmovi

- Ljuske - tela ograničena dvema (paralelnim) površima i cilindričnom površi ortogonalnoj na njih
- Debljina ljuske "h" - konstantno ili promenljivo rastojanje površi
- Srednja površ ljuske - polovi debljinu ljuske
- Kontura ljuske - kriva preseka srednje površi i cilindrične površi koja ograničava ljusku
- Aksijalna smičuća i fleksiona krutost (aksijalno naprezanje, savijanje, smicanje, torzija)

Klasifikacija ljuski prema odnosu debljine i krivine srednje površi

- tanke ljuske ($h/r \leq 1/20$)...
- debele ljuske ($h/r > 1/20$)...

Pretpostavke teorije tankih ljuski

- debljina ljuske je mala u odnosu na ostale dimenzije ljuske (dimenzije osnove, prečnik krivine srednje površi, ...)...
- pravolinijski element koji je normalan na srednju površ, posle deformacije ostaje prav, normalan na srednju površ i ne menja dužinu...
- ugibi su mali u odnosu na debljinu ljuske (ne važi za veoma tanke ljuske)...
- naponi normalni na srednju površ mogu da se zanemare u odnosu na ostale komponentalne napone...

Definisanje srednje površi ljuske

$$\vec{r} = \vec{r}(\alpha, \beta)$$

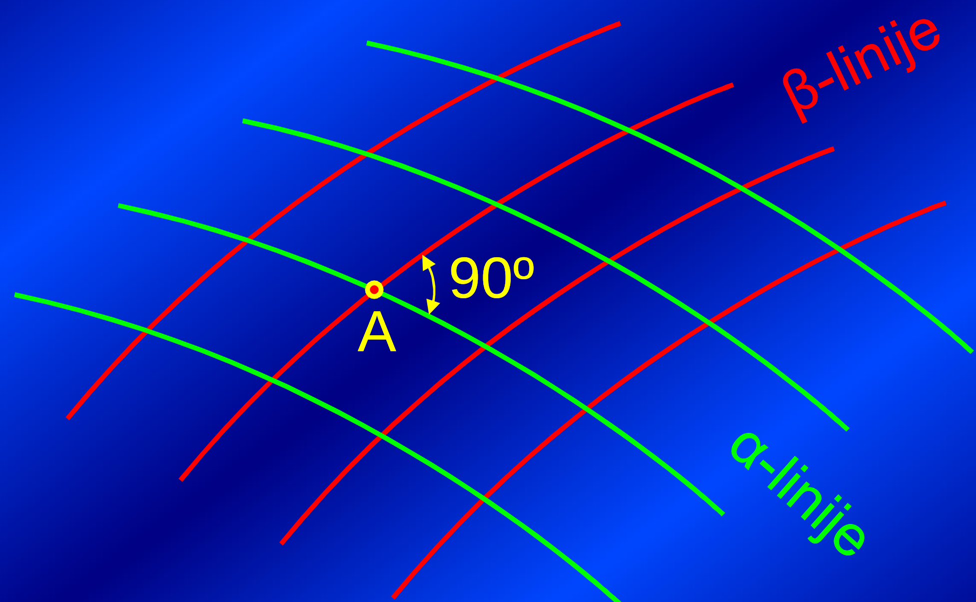
$$x = x(\alpha, \beta)$$

$$y = y(\alpha, \beta)$$

$$z = z(\alpha, \beta)$$

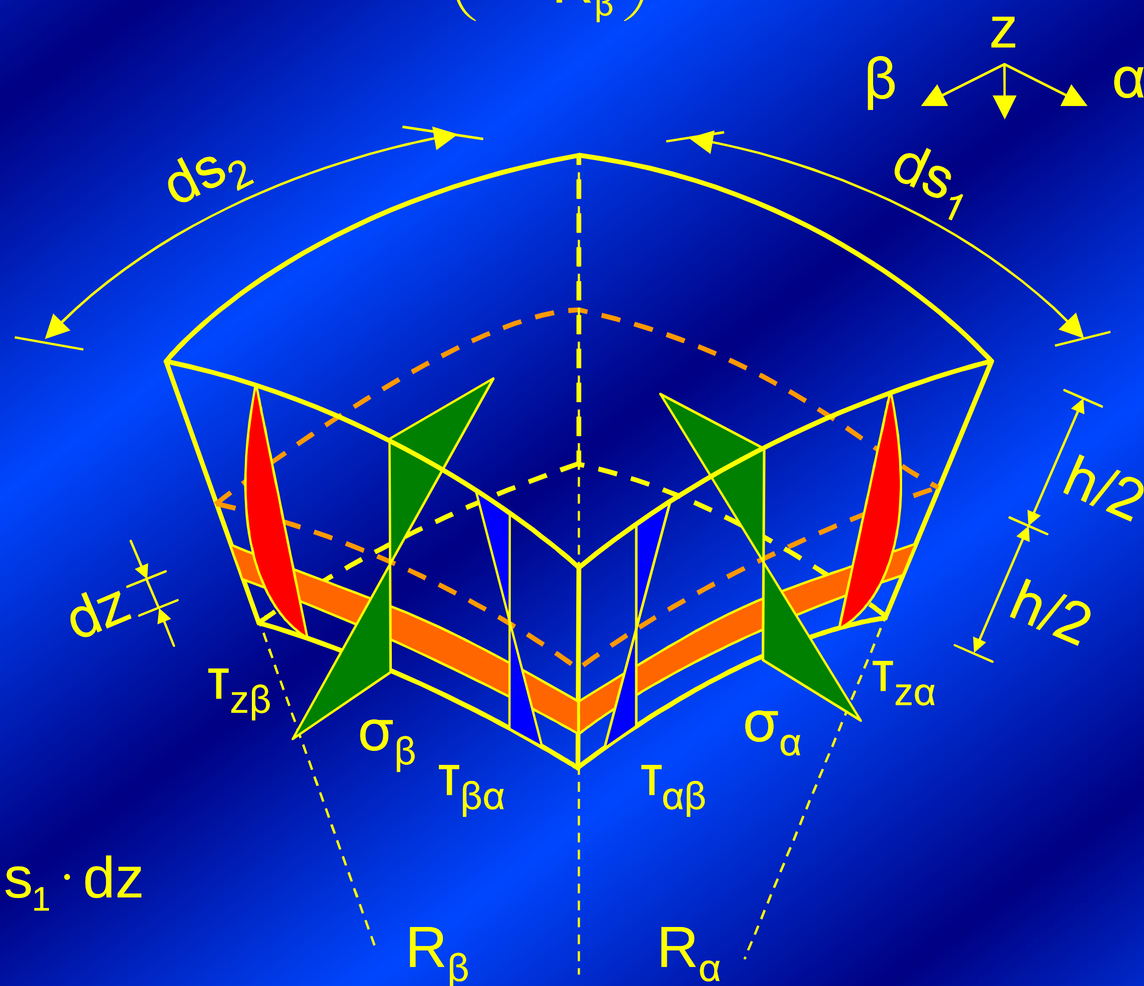
$$F(x, y, z) = 0$$

$$z = f(x, y)$$



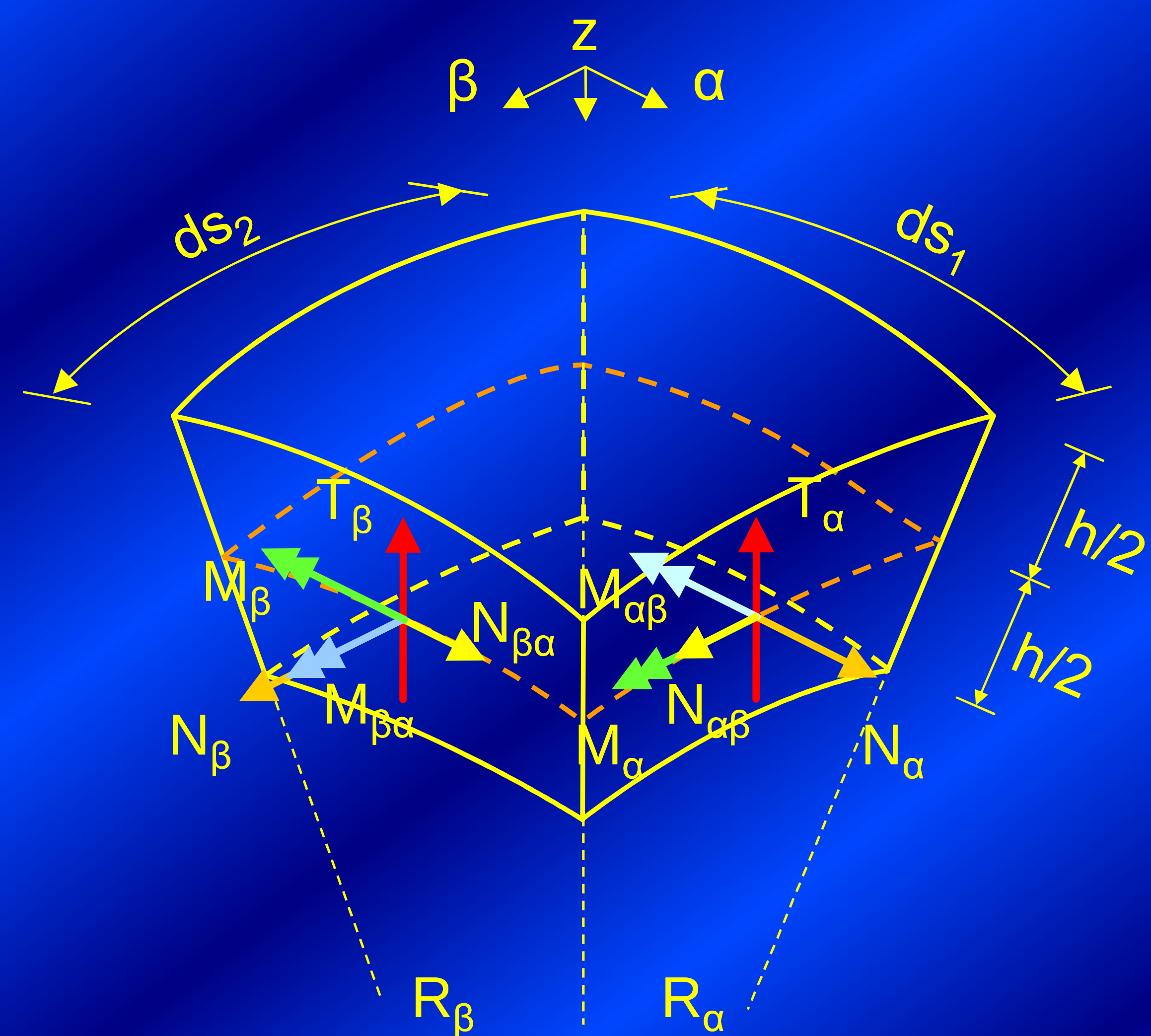
Komponentalni naponi

$$dF_\alpha = \left(1 - \frac{z}{R_\beta}\right) \cdot ds_2 \cdot dz$$



$$dF_\beta = \left(1 - \frac{z}{R_\alpha}\right) \cdot ds_1 \cdot dz$$

Sile u presecima



$$M_{\alpha} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{\alpha} \cdot \left(1 - \frac{z}{R_{\beta}}\right) \cdot z \cdot dz \qquad M_{\beta} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{\beta} \cdot \left(1 - \frac{z}{R_{\alpha}}\right) \cdot z \cdot dz$$

$$T_{\alpha z} = T_{\alpha} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \tau_{\alpha z} \cdot \left(1 - \frac{z}{R_{\beta}}\right) \cdot dz \qquad T_{\beta z} = T_{\beta} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \tau_{\beta z} \cdot \left(1 - \frac{z}{R_{\alpha}}\right) \cdot dz$$

$$M_{\alpha\beta} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \tau_{\alpha\beta} \cdot \left(1 - \frac{z}{R_{\beta}}\right) \cdot z \cdot dz \qquad M_{\beta\alpha} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \tau_{\beta\alpha} \cdot \left(1 - \frac{z}{R_{\alpha}}\right) \cdot z \cdot dz$$

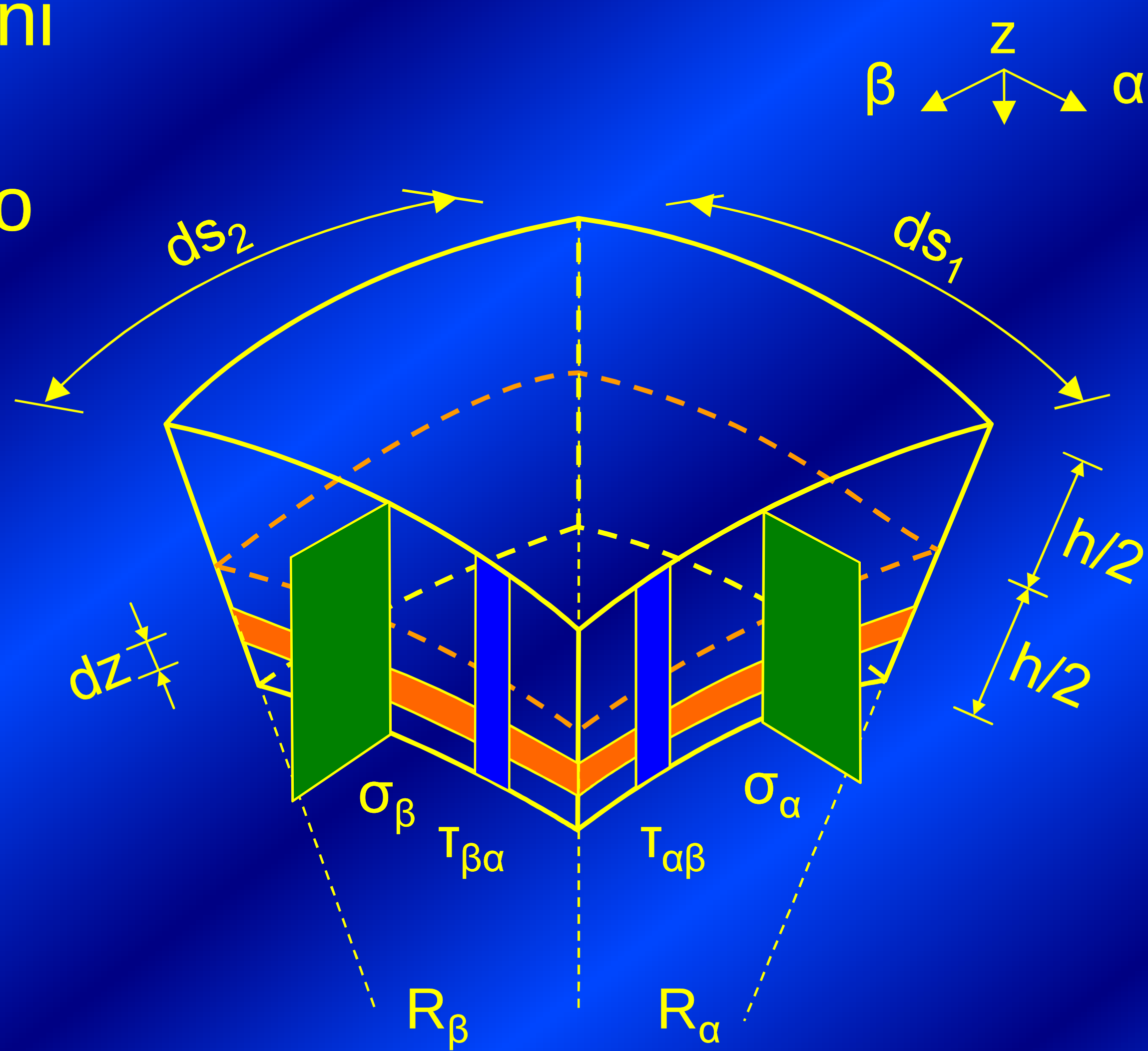
$$N_{\alpha} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{\alpha} \cdot \left(1 - \frac{z}{R_{\beta}}\right) \cdot dz \qquad N_{\beta} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{\beta} \cdot \left(1 - \frac{z}{R_{\alpha}}\right) \cdot dz$$

$$N_{\alpha\beta} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \tau_{\alpha\beta} \cdot \left(1 - \frac{z}{R_{\beta}}\right) \cdot dz \qquad N_{\beta\alpha} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \tau_{\beta\alpha} \cdot \left(1 - \frac{z}{R_{\alpha}}\right) \cdot dz$$

Membranska (bezmomentna) teorija ljuski

- zanemarljiva fleksiona krutost - - mala debljina ljuske (h/R<<1)
- odgovarajući oblik (topologija) ljuske (glatka srednja površ)
- odgovarajući način oslanjanja ljuske (samo membranske N_{α}, N_{β}, $N_{\alpha\beta}$ i $N_{\beta\alpha}$ sile na konturi)
- odgovarajuća konfiguracija opterećenja (bez naglih promena)

Komponentalni naponi (membransko stanje)

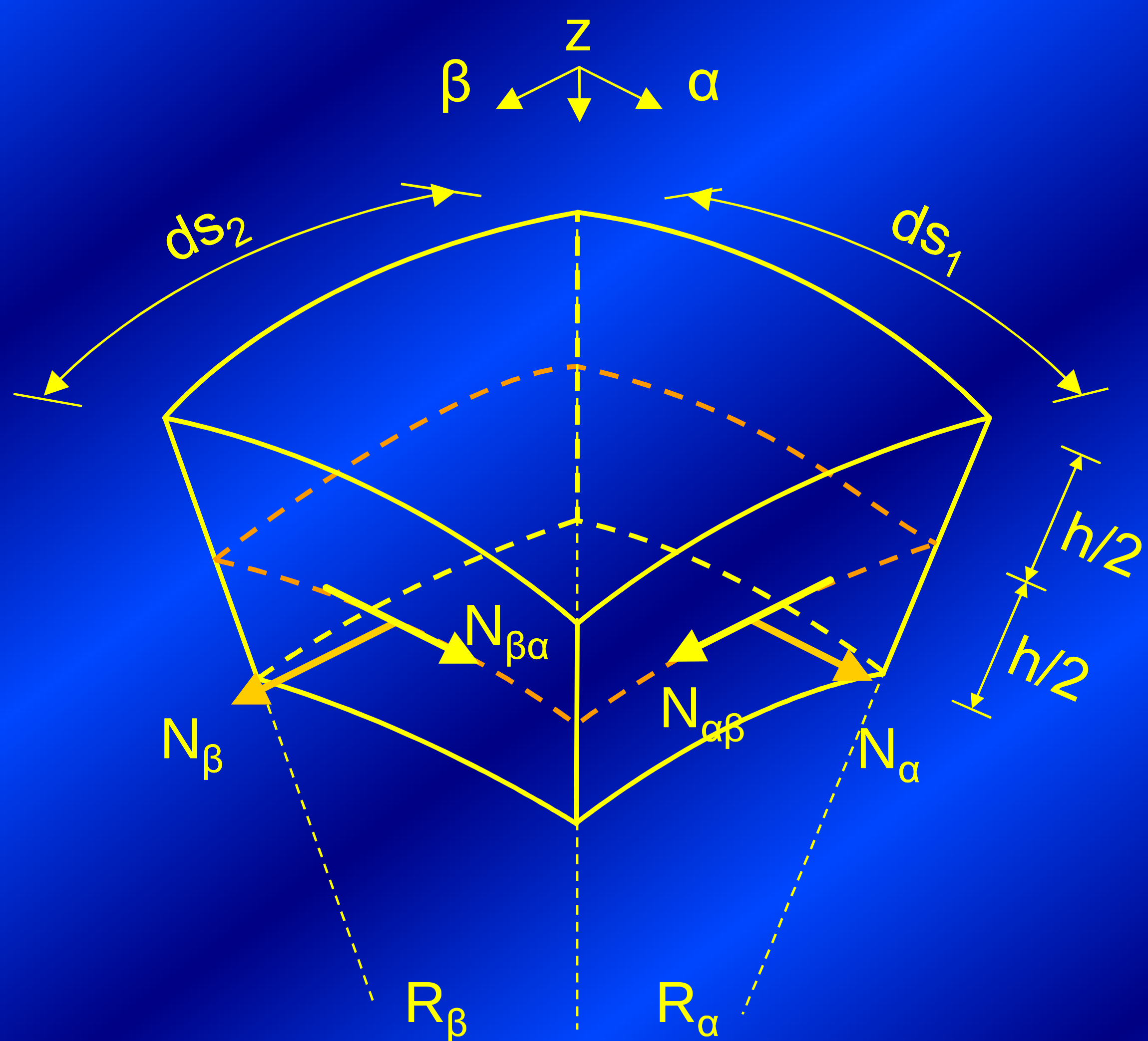


$$N_\alpha = \sigma_\alpha \cdot h$$

$$N_\beta = \sigma_\beta \cdot h$$

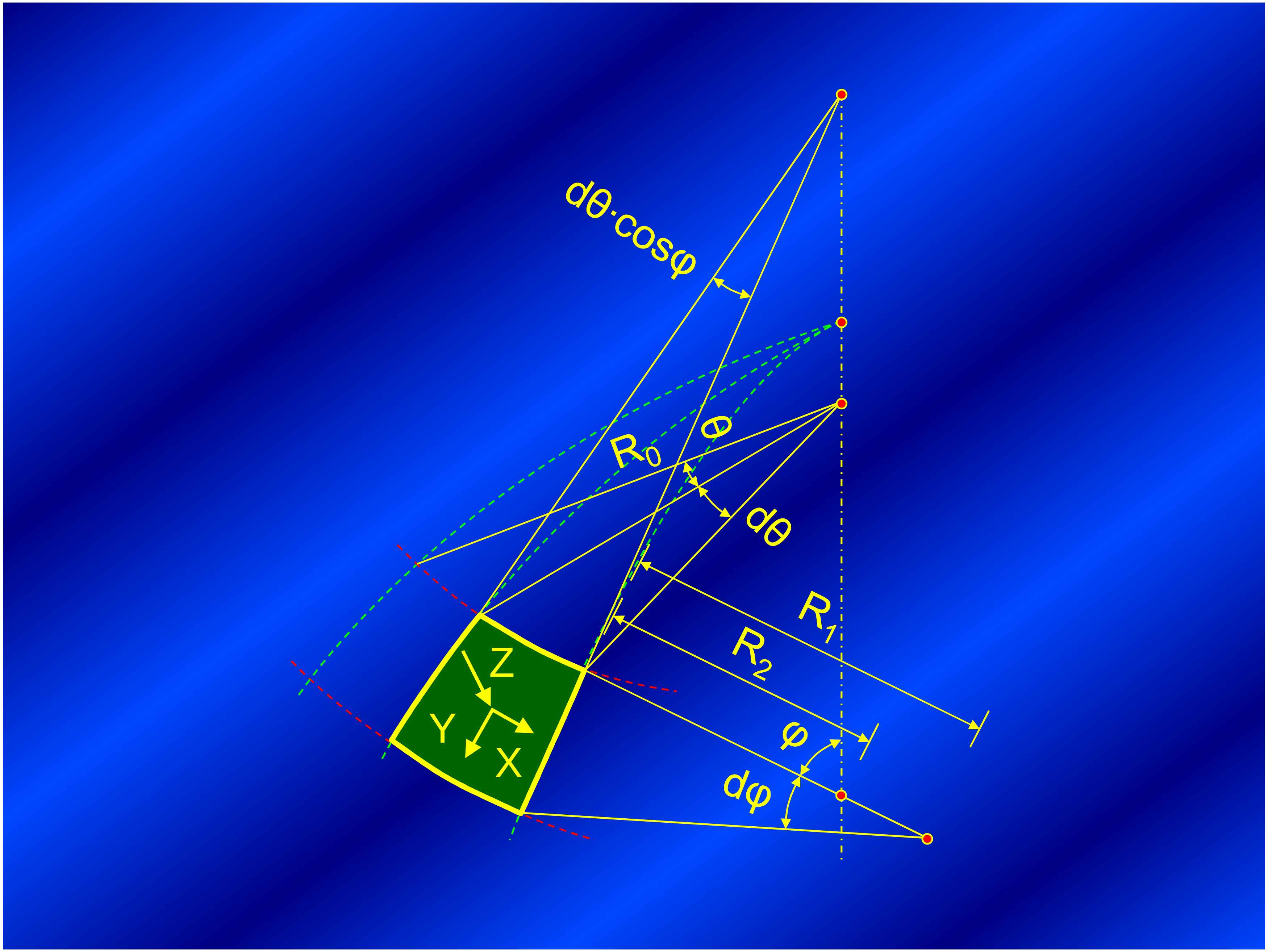
$$N_{\alpha\beta} = N_{\beta\alpha} = \tau_{\alpha\beta} \cdot h$$

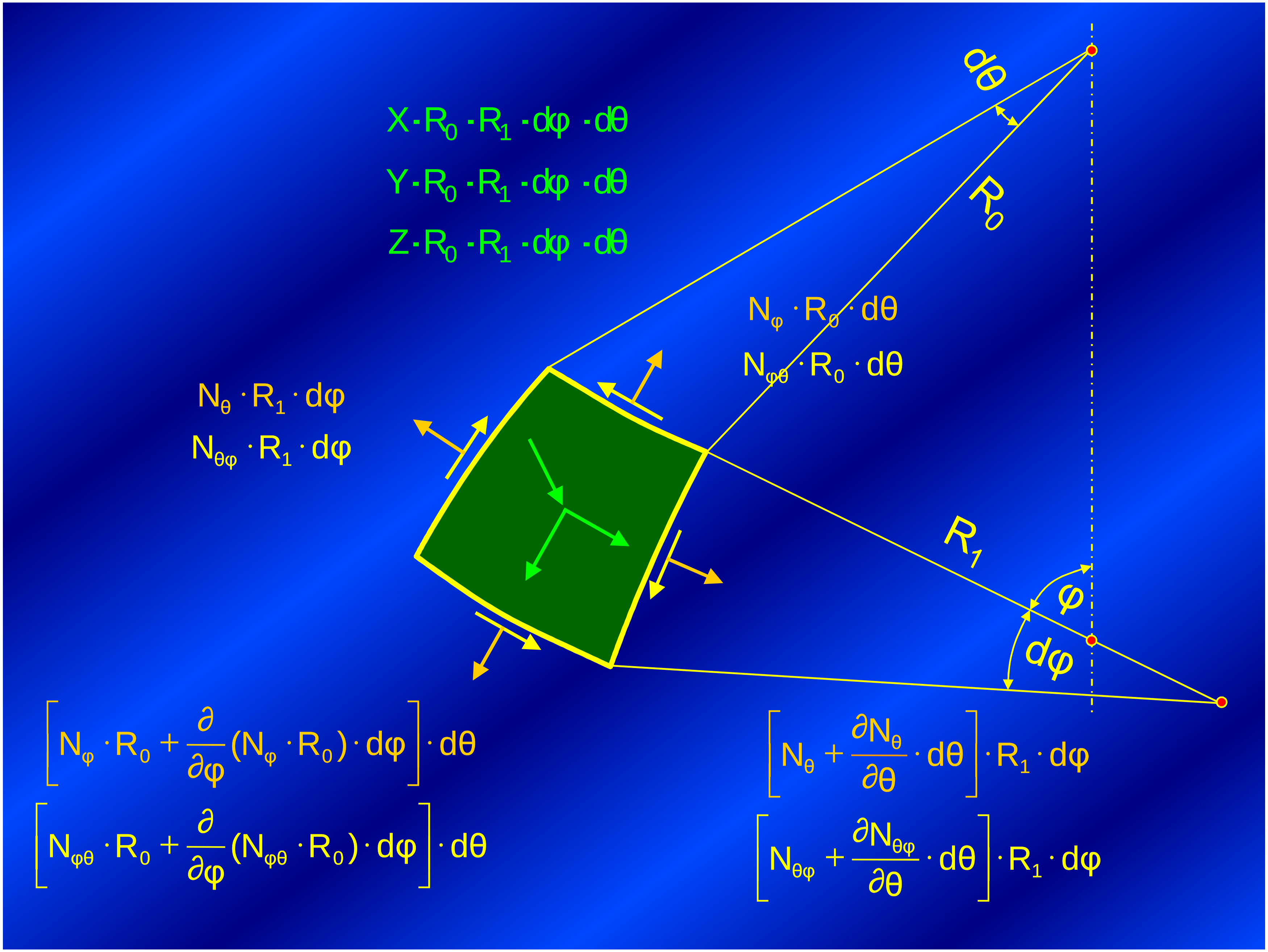
Sile u presecima



Membranska (bezmomentna) teorija rotacionih ljuski

- srednja površ rotacione ljuske je rotaciona površ - nastaje rotacijom ravne krive oko neke ose (ose ljuske)
- α- i β-linije rotacione ljuske su meridijani i paralele (preseci srednje površi sa ravnima koje sadrže osu i ravnima normalnim na osu ljuske)
- krivolinijske koordinate su: "φ" - ugao između ose ljuske i normale na srednju površ i "θ" - ugao položaja na paraleli
- R_1 i R_2 - radijusi krivine meridijana


dθ·cosφ
θ
R0
dθ
R1
R2
Z
Y
X
φ
dφ



$X \cdot R_0 \cdot R_1 \cdot d\varphi \cdot d\theta$
$Y \cdot R_0 \cdot R_1 \cdot d\varphi \cdot d\theta$
$Z \cdot R_0 \cdot R_1 \cdot d\varphi \cdot d\theta$
$d\theta$
R_0
$N_\varphi \cdot R_0 \cdot d\theta$
$N_{\varphi\theta} \cdot R_0 \cdot d\theta$
$N_\theta \cdot R_1 \cdot d\varphi$
$N_{\theta\varphi} \cdot R_1 \cdot d\varphi$
R_1
φ
$d\varphi$
$\left[N_\varphi \cdot R_0 + \frac{\partial}{\partial\varphi}(N_\varphi \cdot R_0) \cdot d\varphi\right] \cdot d\theta$
$\left[N_{\varphi\theta} \cdot R_0 + \frac{\partial}{\partial\varphi}(N_{\varphi\theta} \cdot R_0) \cdot d\varphi\right] \cdot d\theta$
$\left[N_\theta + \frac{\partial N_\theta}{\partial\theta} \cdot d\theta\right] \cdot R_1 \cdot d\varphi$
$\left[N_{\theta\varphi} + \frac{\partial N_{\theta\varphi}}{\partial\theta} \cdot d\theta\right] \cdot R_1 \cdot d\varphi$

Uslovi ravnoteže rotacionih ljuski

$$\frac{\partial N_\theta}{\partial \theta}\cdot R_1+\frac{\partial}{\partial \varphi}(N_{\varphi\theta}\cdot R_0)+N_{\theta\varphi}\cdot R_1\cdot\cos\varphi+X\cdot R_0\cdot R_1=0$$

$$\frac{\partial}{\partial \varphi}(N_\varphi\cdot R_0)+\frac{\partial N_{\varphi\theta}}{\partial \theta}\cdot R_1-N_\theta\cdot R_1\cdot\cos\varphi+Y\cdot R_0\cdot R_1=0$$

$$\frac{N_\varphi}{R_1}+\frac{N_\theta}{R_2}+Z=0$$

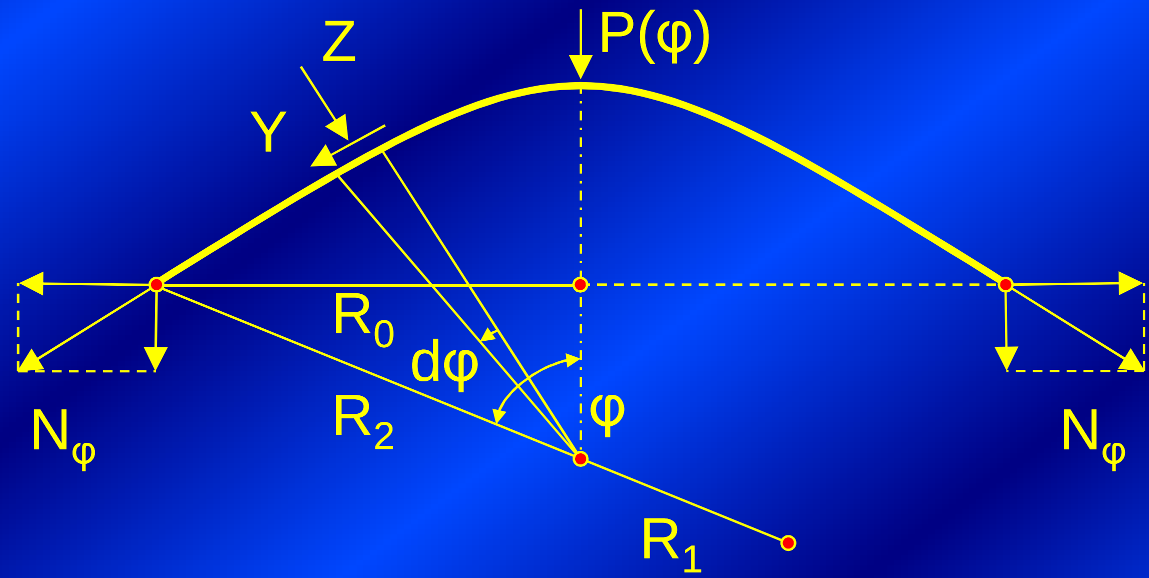
Rotaciono simetrično opterećenje

$$X = 0 \qquad N_{\varphi\theta} = N_{\theta\varphi} = 0$$

$$\frac{d}{d\varphi}(N_\varphi \cdot R_0) - N_\theta \cdot R_1 \cdot \cos\varphi + Y \cdot R_0 \cdot R_1 = 0$$

$$\frac{N_\varphi}{R_1} + \frac{N_\theta}{R_2} + Z = 0 \qquad \Rightarrow \qquad N_\theta = R_2 \cdot \left(Z + \frac{N_\varphi}{R_1} \right)$$

$$N_\varphi = -\frac{1}{R_0 \cdot \sin\varphi} \cdot \left[\int_0^\varphi R_1 \cdot R_0 \cdot (Y \cdot \sin\varphi + Z \cdot \cos\varphi) \cdot d\varphi + C \right]$$

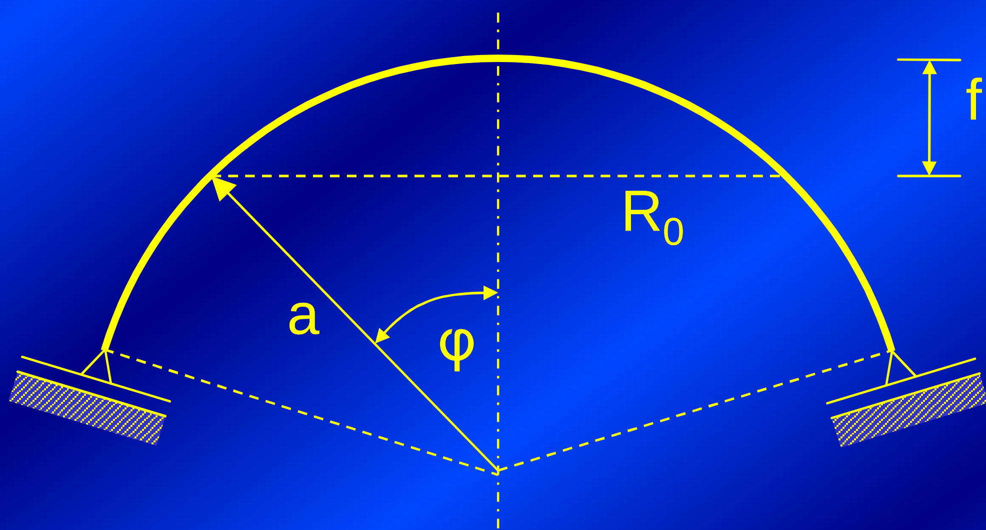


$$\sum V = 0$$

$$2 \cdot \pi \cdot R_0 \cdot N_\varphi \cdot \sin\varphi + P(\varphi) = 0$$

$$N_\varphi = -\frac{P(\varphi)}{2 \cdot \pi \cdot R_0 \cdot \sin\varphi}$$

Sferna kupola



$$R_1 = R_2 = a$$

$$R_0 = a \cdot \sin\varphi$$

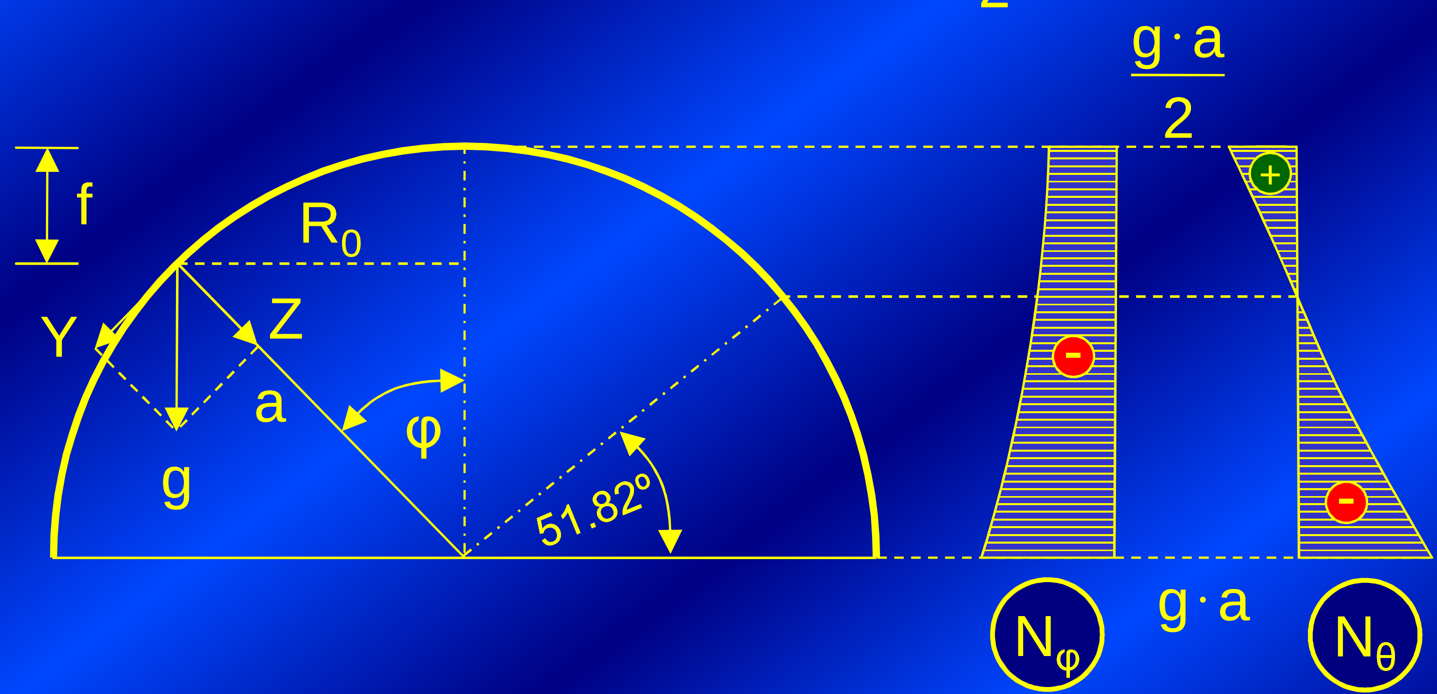
$$Y = g \cdot \sin\varphi$$

$$Z = g \cdot \cos\varphi$$

$$P(\varphi) = 2 \cdot \pi \cdot g \cdot a^2 \cdot \int_0^{\varphi} \sin\varphi \cdot d\varphi = 2 \cdot \pi \cdot g \cdot a^2 \cdot (1 - \cos\varphi)$$

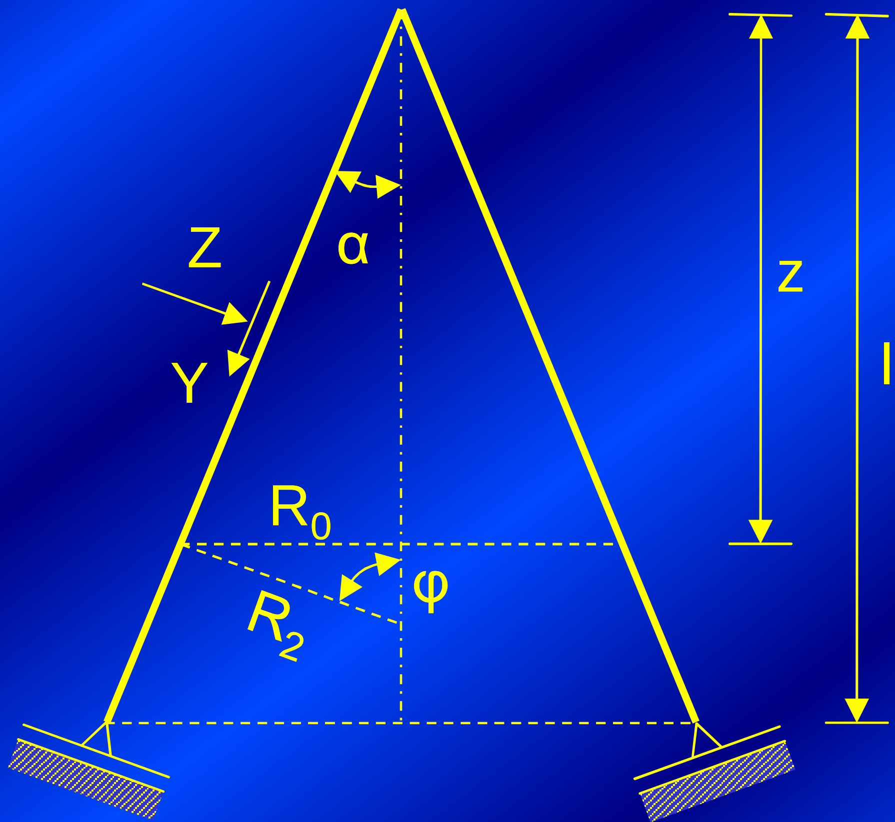
$$N_\varphi = -\frac{g \cdot a}{1 + \cos\varphi} \qquad N_\theta = -g \cdot a \cdot \left(\cos\varphi - \frac{1}{1 + \cos\varphi} \right)$$

$$\varphi = 0 \qquad N_\varphi = N_\theta = -\frac{g \cdot a}{2}$$



$$\varphi = \frac{\pi}{2} \qquad N_\varphi = -N_\theta = -g \cdot a$$

Konusna ljuska



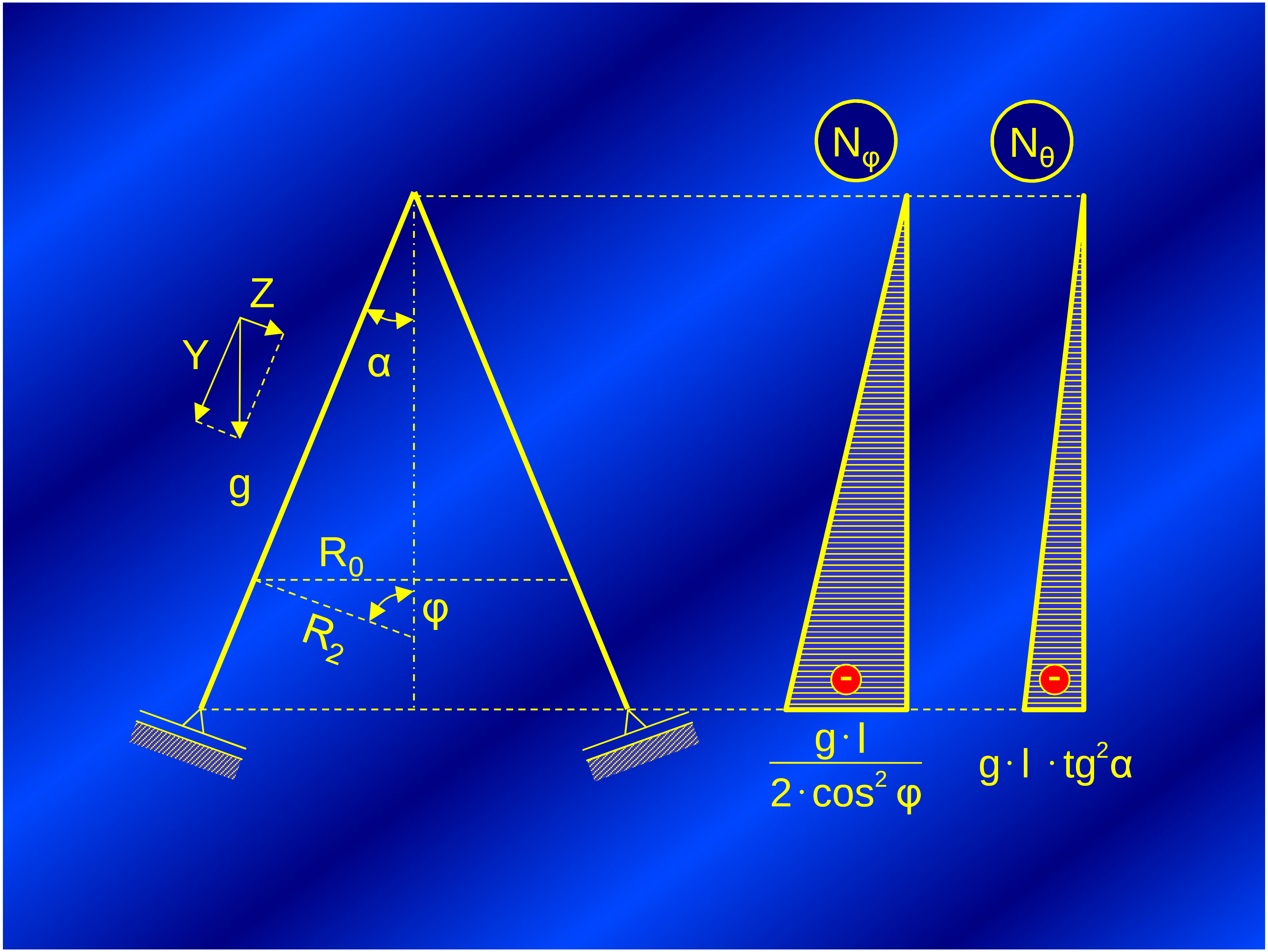
$$R_1 = \infty \qquad R_2 = \frac{R_0}{\cos\alpha}$$

$$R_0 = Z \cdot \mathrm{tg}\alpha$$

$$R_1 \cdot d\varphi = \frac{dz}{\cos\alpha}$$

$$Y \cdot \cos\alpha + Z \cdot \sin\alpha = g$$

$$P(z) = \pi \cdot g \cdot z^2 \cdot \frac{\mathrm{tg}\alpha}{\cos\alpha} \qquad N_\varphi = -\frac{g \cdot z}{2 \cdot \cos^2\varphi} \qquad N_\theta = -g \cdot z \cdot \mathrm{tg}^2\alpha$$

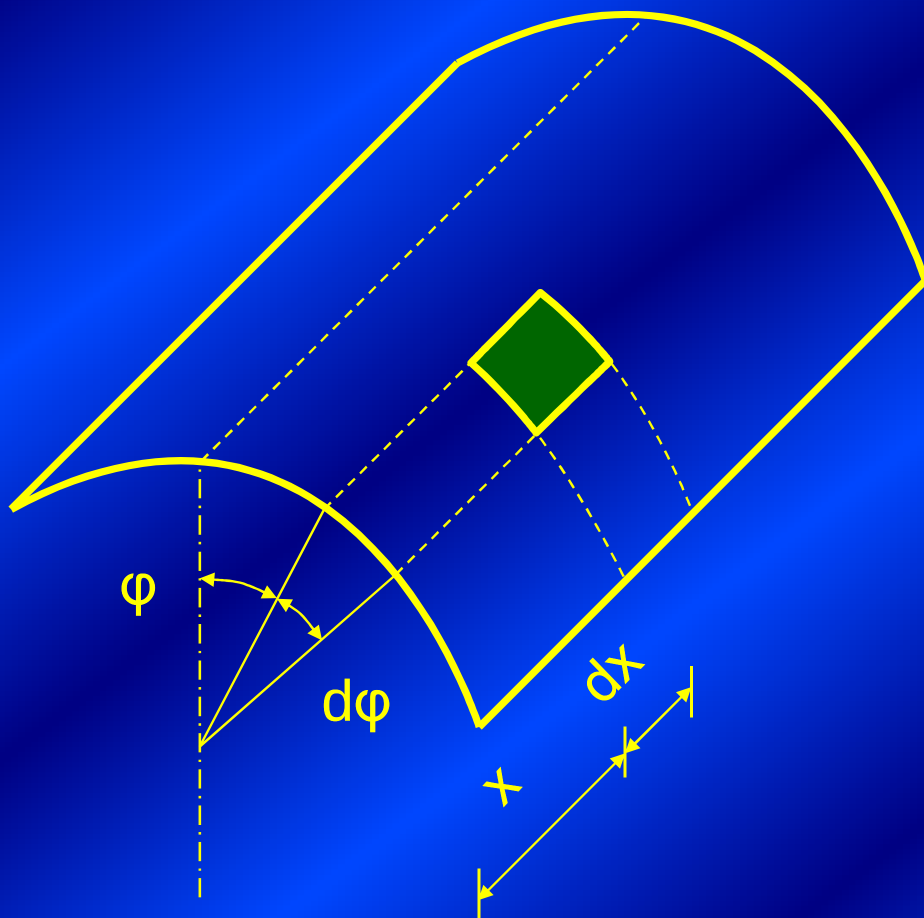


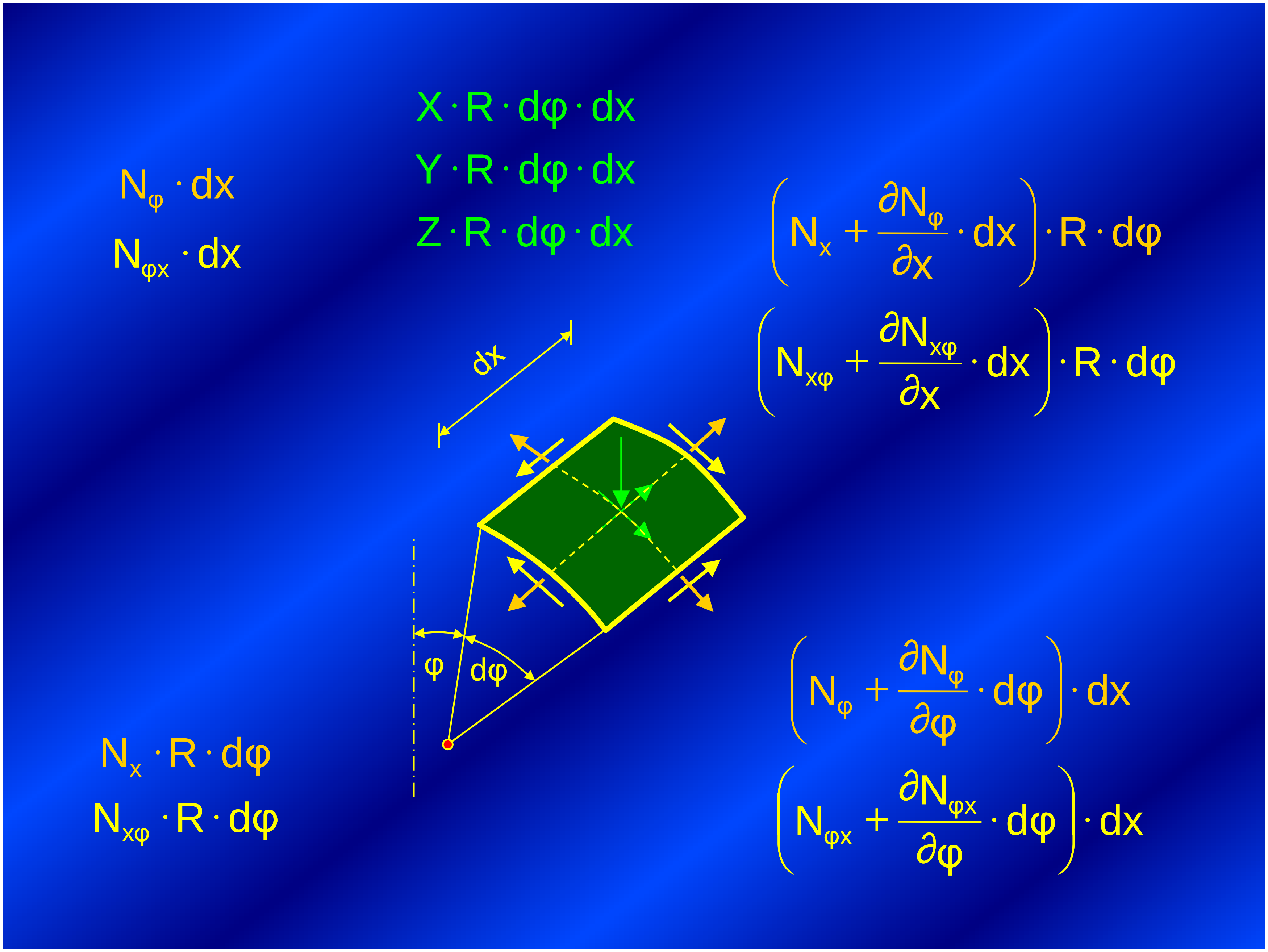
Z
Y
α
g
R_0
φ
R_2
N_φ
N_θ
-
-
$\frac{g \cdot l}{2 \cdot \cos^2 \varphi}$
$g \cdot l \cdot tg^2\alpha$

Cilindrična ljuska

koordinate:
x, φ

sile u presecima:
N_x, $N_φ$ i $N_{xφ}$




$X \cdot R \cdot d\varphi \cdot dx$
$Y \cdot R \cdot d\varphi \cdot dx$
$Z \cdot R \cdot d\varphi \cdot dx$
$N_\varphi \cdot dx$
$N_{\varphi x} \cdot dx$
$\left(N_x + \frac{\partial N_\varphi}{\partial x} \cdot dx\right) \cdot R \cdot d\varphi$
$\left(N_{x\varphi} + \frac{\partial N_{x\varphi}}{\partial x} \cdot dx\right) \cdot R \cdot d\varphi$
dx
φ
dφ
$\left(N_\varphi + \frac{\partial N_\varphi}{\partial \varphi} \cdot d\varphi\right) \cdot dx$
$\left(N_{\varphi x} + \frac{\partial N_{\varphi x}}{\partial \varphi} \cdot d\varphi\right) \cdot dx$
$N_x \cdot R \cdot d\varphi$
$N_{x\varphi} \cdot R \cdot d\varphi$

Uslovi ravnoteže

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{x\varphi}}{R \cdot \partial \varphi} + X = 0 \qquad \frac{\partial N_\varphi}{R \cdot \partial \varphi} + \frac{\partial N_{\varphi x}}{\partial x} + Y = 0 \qquad \frac{N_\varphi}{R} + Z = 0$$

$$N_\varphi = -Z \cdot R \qquad N_{\varphi x} = -\int \left(Y + \frac{\partial N_\varphi}{R \cdot \partial \varphi} \right) dx + C_1(\varphi)$$

$$N_x = -\int \left(X + \frac{\partial N_{\varphi x}}{R \cdot \partial \varphi} \right) dx + C_2(\varphi)$$

Kružna cilindrična ljuska

$$X = \sum_{n=0}^{\infty} X_n \cdot \cos n\varphi \qquad Y = \sum_{n=0}^{\infty} Y_n \cdot \cos n\varphi \qquad Z = \sum_{n=0}^{\infty} Z_n \cdot \cos n\varphi$$

$$N_\varphi = -Z_n \cdot a \cdot \cos n\varphi$$

$$N_{\varphi x} = -\sin n\varphi \cdot \int (Y_n + n \cdot Z_n) \cdot dx + C_1(\varphi)$$

$$\frac{1}{a} \cdot \frac{\partial N_{\varphi x}}{\partial \varphi} = -\frac{n}{a} \cdot \cos \varphi \cdot \int (Y_n + n \cdot Z_n) \cdot dx + \frac{dC_1(\varphi)}{a \cdot d\varphi}$$

$$N_x = -\frac{\cos n\varphi}{a} \cdot \int \left[X_n \cdot a - n \cdot \int (Y_n + n \cdot Z_n) \cdot dx \right] \cdot dx - \frac{dC_1(\varphi)}{a \cdot d\varphi} + C_2(\varphi)$$

$$C_1(\varphi) = A_1 \cdot \sin n\varphi \qquad C_2(\varphi) = A_2 \cdot \cos n\varphi$$

$$N_\varphi = Z_n \cdot a \cdot \cos n\varphi$$

$$N_{\varphi x} = [(Y_n + n \cdot Z_n) \cdot x - A_1] \cdot \sin n\varphi$$

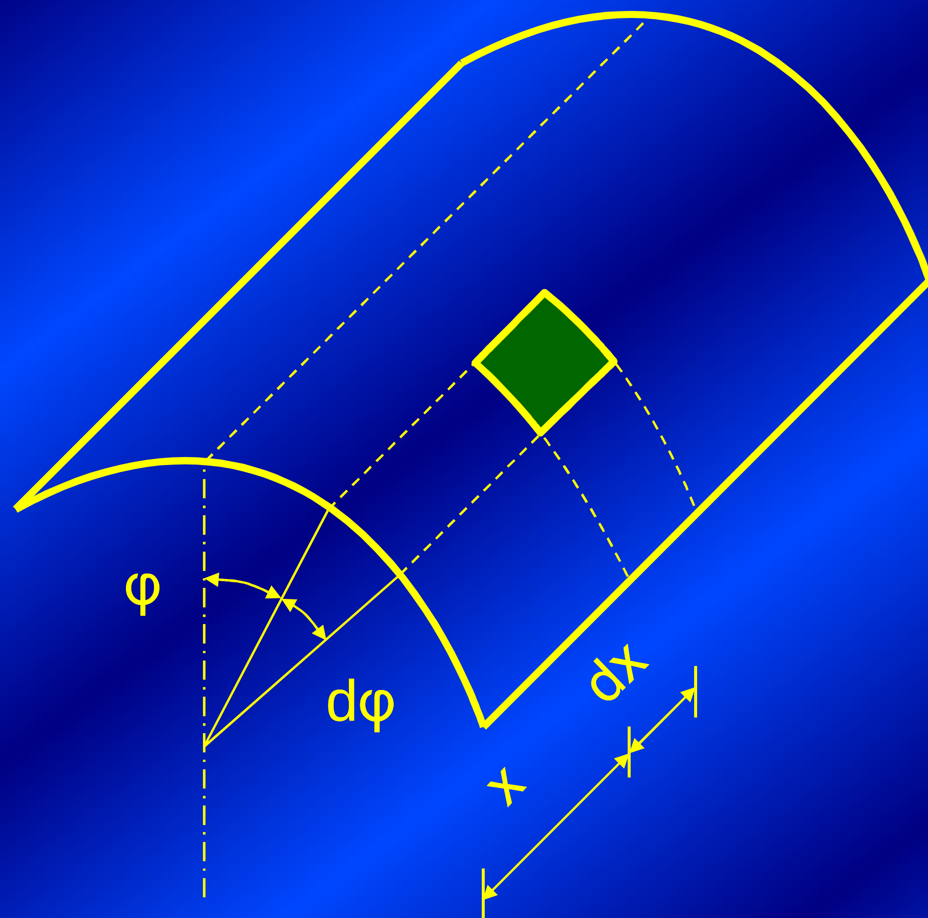
$$N_{\varphi x} = \left\{ \frac{n}{a} \cdot \left[(Y_n + n \cdot Z_n) \cdot \frac{x^2}{2} - A_1 \cdot x \right] + A_2 \right\} \cdot \cos n\varphi$$

Savijanje kružne cilindrične ljuske
- rotaciona simetrija -

koordinate:
x, φ

sile u presecima:
T_x, N_φ, M_x i M_φ

opterećenje:
Z



Uslovi ravnoteže

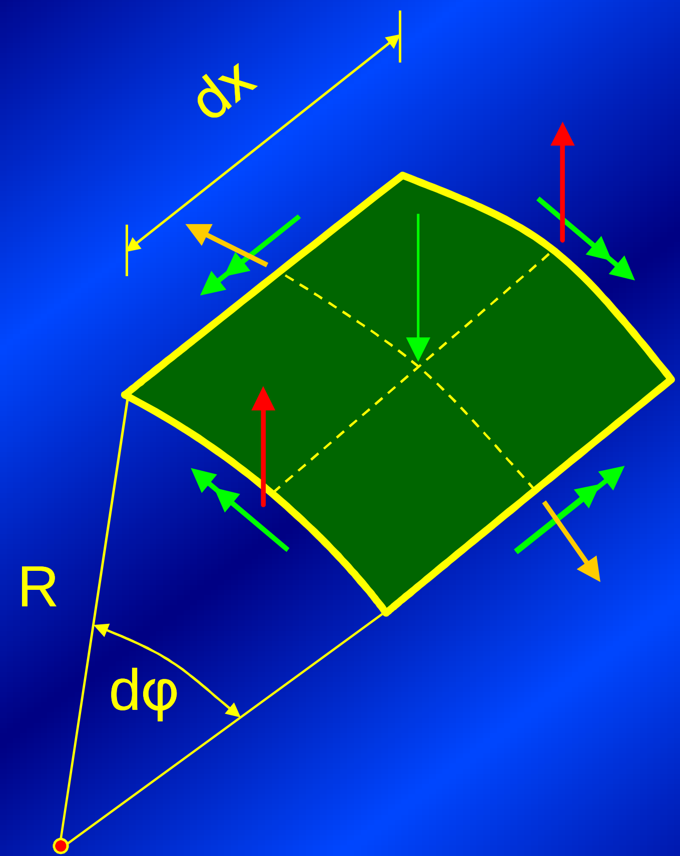
$$Z \cdot R \cdot d\varphi \cdot dx$$

$$N_\varphi \cdot dx$$

$$M_\varphi \cdot dx$$

$$T_x \cdot R \cdot d\varphi$$

$$M_x \cdot R \cdot d\varphi$$



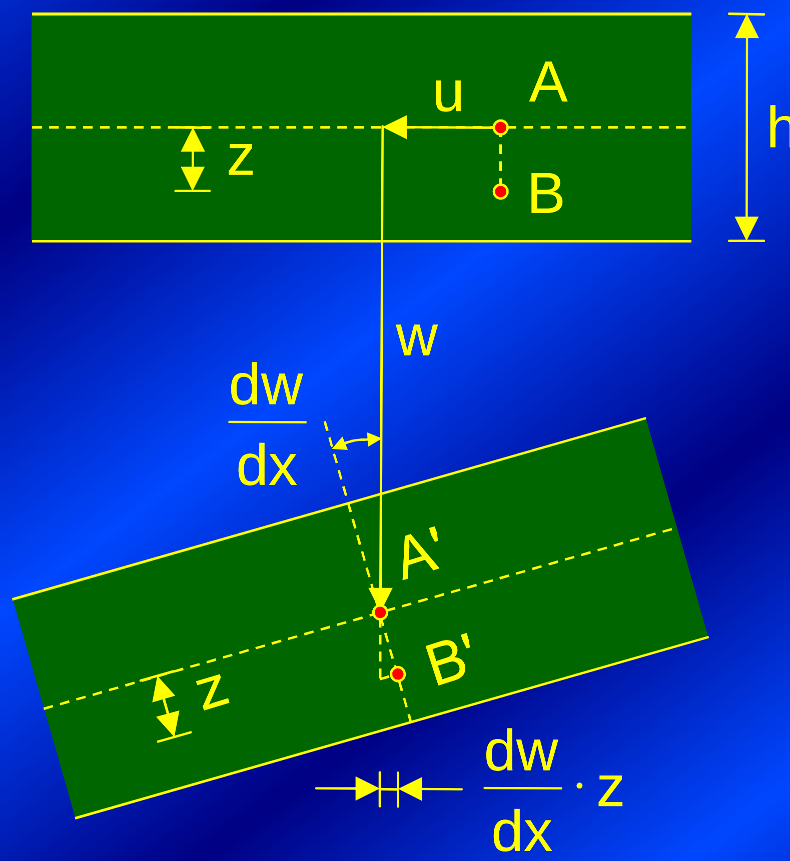
$$N_\varphi \cdot dx$$

$$M_\varphi \cdot dx$$

$$\left(M_x + \frac{dM_x}{dx} \cdot dx \right) \cdot R \cdot d\varphi$$

$$\left(T_x + \frac{dT_x}{dx} \cdot dx \right) \cdot R \cdot d\varphi$$

$$N_{\varphi x} = N_{x\varphi} = 0 \qquad M_{\varphi x} = M_{x\varphi} = 0 \qquad T_{\varphi} = 0 \qquad N_x = \frac{\sum X}{2 \cdot R \cdot \pi}$$



$$\frac{dM_x}{dx} - T_x = 0$$

$$\frac{N_{\varphi}}{R} + \frac{dT_x}{dx} + Z = 0$$

$$\varepsilon_x = \frac{du}{dx} \qquad \varepsilon_x = -\frac{w}{R}$$

$$u_z = u - \frac{dw}{dx} \cdot z$$

$$\varepsilon_{x_z} = \frac{du_z}{dx} = -\frac{d^2w}{dx^2} \cdot z + \frac{du}{dx}$$

$$w_z = w \qquad \Rightarrow \qquad \varepsilon_{\varphi_z} = -\frac{w}{a}$$

$$\sigma_x = \frac{E}{1-v^2} \cdot (\varepsilon_{x_z} + v \cdot \varepsilon_{\varphi_z}) = \frac{E}{1-v^2} \cdot \left(\frac{du}{dx} - \frac{d^2w}{dx^2} \cdot z - v \cdot \frac{w}{a} \right)$$

$$\sigma_\varphi = \frac{E}{1-v^2} \cdot (\varepsilon_{\varphi_z} + v \cdot \varepsilon_{x_z}) = \frac{E}{1-v^2} \cdot \left(v \cdot \frac{du}{dx} - v \cdot \frac{d^2w}{dx^2} \cdot z - \frac{w}{a} \right)$$

$$M_x = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_x \cdot z \cdot dz = \frac{E \cdot h^3}{12 \cdot (1 - v^2)} \cdot \frac{d^2 w}{dx^2} = k \cdot \frac{d^2 w}{dx^2}$$

$$N_x = \frac{E \cdot h}{1 - v^2} \cdot \left(\frac{du}{dx} - v \cdot \frac{w}{a} \right) = 0 \quad \Rightarrow \quad \frac{du}{dx} = v \cdot \frac{w}{a}$$

$$N_\varphi = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_\varphi \cdot dz = \frac{E}{1 - v^2} \cdot \left(v \cdot \frac{du}{dx} - \frac{w}{a} \right) = -E \cdot h \cdot \frac{w}{a}$$

$$T_x = \frac{dM_x}{dx} \qquad \frac{d^2 M_x}{dx^2} + \frac{N_\varphi}{a} = -Z$$

$$\frac{d^2}{dx^2}\left(k \cdot \frac{d^2 w}{dx^2}\right) + E \cdot h \cdot \frac{w}{a^2} = Z$$

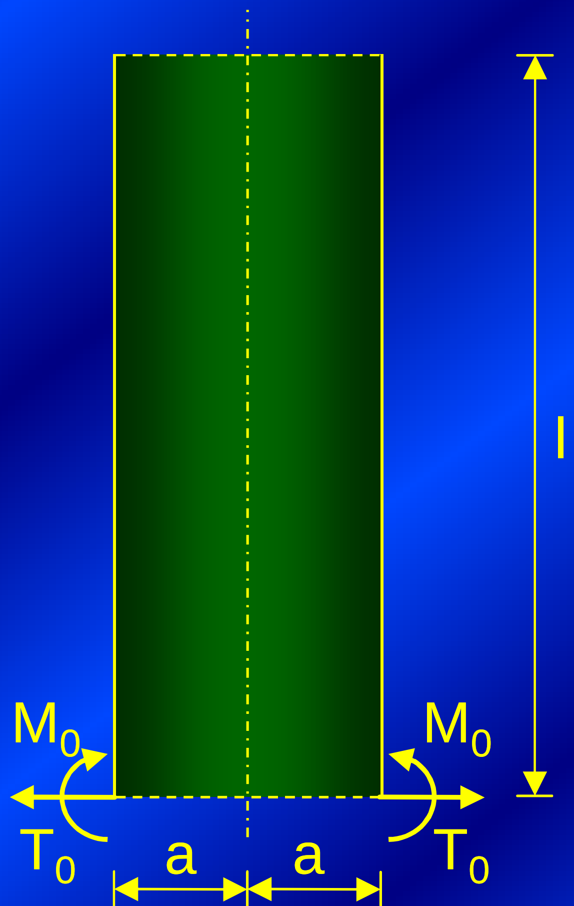
$$\frac{d^4 w}{dx^4} + \frac{12 \cdot (1 - v^2)}{a^2 \cdot h^2} \cdot w = \frac{Z}{k} \qquad \beta^4 = \frac{3 \cdot (1 - v^2)}{a^2 \cdot h^2}$$

$$\frac{d^4 w}{dx^4} + 4 \cdot \beta^4 \cdot w = \frac{Z}{k}$$

$$w = w_0 + e^{\beta \cdot x} \cdot (C_1 \cdot \cos \beta x + C_2 \cdot \sin \beta x) +$$

$$+ e^{-\beta \cdot x} \cdot (C_3 \cdot \cos \beta x + C_4 \cdot \sin \beta x)$$

Duga cilindrična ljuska



za x □ w=0 $C_1=C_2=0$

$$w = e^{-\beta \cdot x} \cdot (C_3 \cdot \cos\beta x + C_4 \cdot \sin\beta x)$$

za x=0

$$M_x = -k \cdot \frac{d^2w}{dx^2} = M_0$$

$$T_x = -k \cdot \frac{d^3w}{dx^3} = T_0$$

$$\frac{dw}{dx} = \beta \cdot e^{-\beta \cdot x} \cdot [(C_1 - C_3) \cdot \cos\beta x - (C_3 + C_4) \cdot \sin\beta x]$$

$$\frac{d^2w}{dx^2} = 2 \cdot \beta^2 \cdot e^{-\beta \cdot x} \cdot (C_3 \cdot \sin\beta x - C_4 \cdot \cos\beta x)$$

$$\frac{d^3w}{dx^3} = 2 \cdot \beta^3 \cdot e^{-\beta \cdot x} \cdot [(C_3 + C_4) \cdot \cos\beta x - (C_3 - C_4) \cdot \sin\beta x]$$

$$M_0 = 2 \cdot \beta^2 \cdot k \cdot C_4 \qquad T_0 = -2 \cdot \beta^3 \cdot k \cdot (C_3 + C_4)$$

$$C_3 = \frac{T_0 + \beta \cdot M_0}{2 \cdot \beta^3 \cdot k} \qquad C_4 = \frac{\beta \cdot M_0}{2 \cdot \beta^3 \cdot k}$$

$$w = -\frac{e^{-\beta\cdot x}}{2\cdot\beta^3\cdot k}\cdot[(T_0+\beta\cdot M_0)\cdot\cos\beta x-\beta\cdot M_0\cdot\sin\beta x]$$

$$\varphi(\alpha) = e^{-\alpha}\cdot(\cos\alpha+\sin\alpha)$$

$$\psi(\alpha) = e^{-\alpha}\cdot(\cos\alpha-\sin\alpha)$$

$$\theta(\alpha) = e^{-\alpha}\cdot\cos\alpha$$

$$\zeta(\alpha) = e^{-\alpha}\cdot\sin\alpha$$

$$\varphi'(\alpha) = -2\cdot\zeta(\alpha)$$

$$\psi'(\alpha) = -2\cdot\theta(\alpha)$$

$$\theta'(\alpha) = -\varphi(\alpha)$$

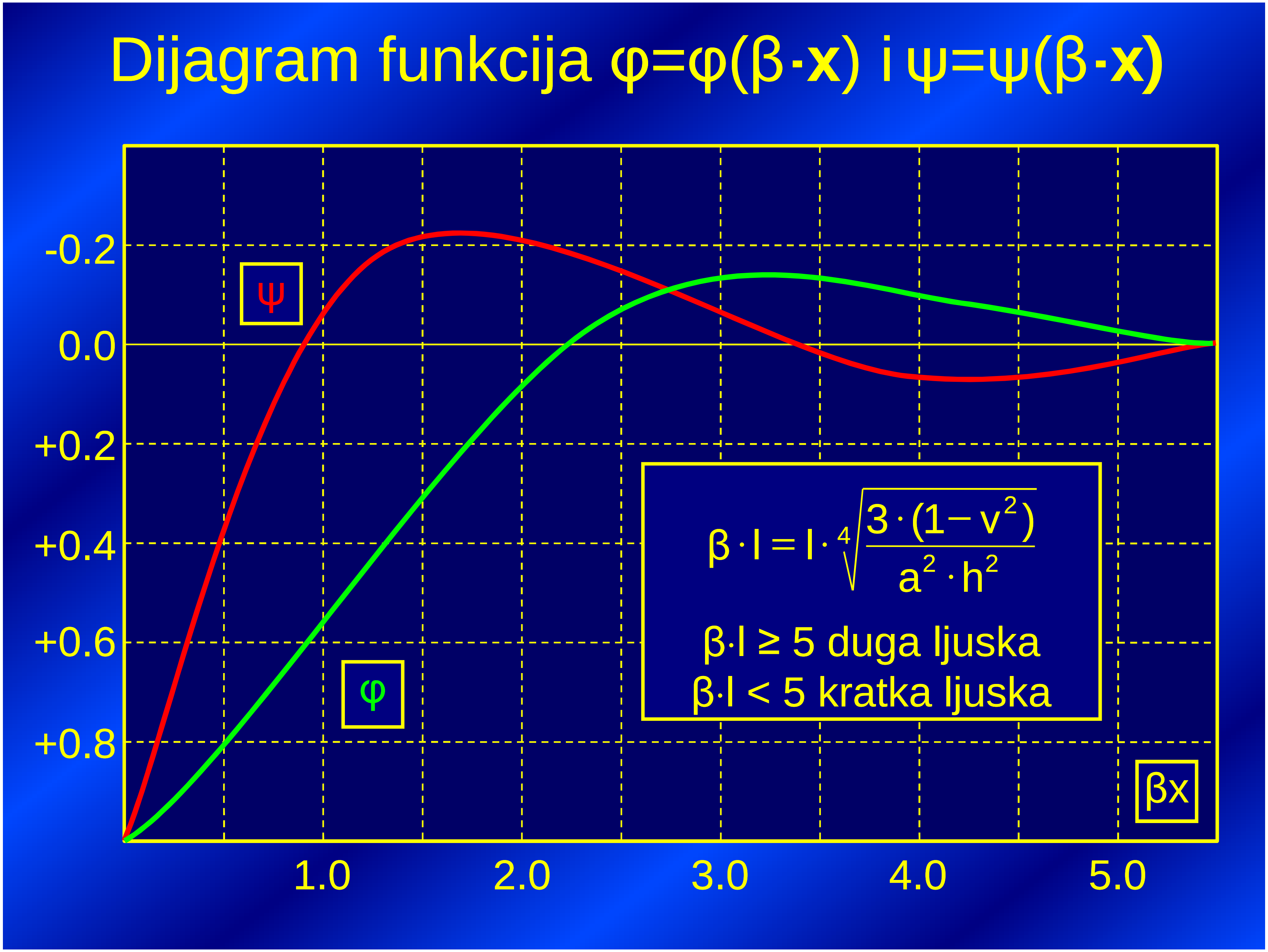
$$\zeta'(\alpha) = \psi(\alpha)$$

$$w = -\frac{1}{2 \cdot \beta^3 \cdot k} \cdot [\beta \cdot M_0 \cdot \psi(\beta x) + T_0 \cdot \theta(\beta x)]$$

$$\frac{dw}{dx} = \frac{1}{2 \cdot \beta^2 \cdot k} \cdot [2 \cdot \beta \cdot M_0 \cdot \theta(\beta x) + T_0 \cdot \varphi(\beta x)]$$

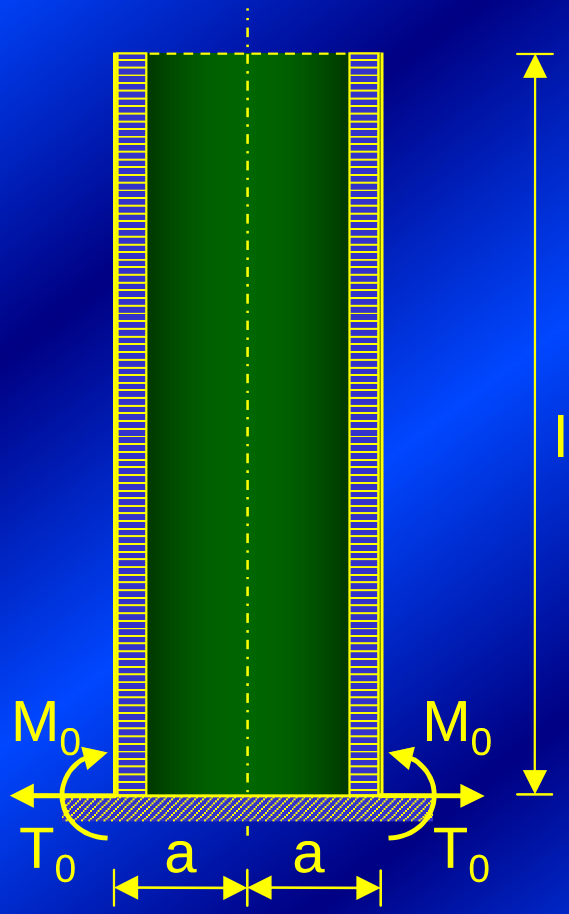
$$\frac{d^2w}{dx^2} = -\frac{1}{\beta \cdot k} \cdot [\beta \cdot M_0 \cdot \varphi(\beta x) + T_0 \cdot \zeta(\beta x)]$$

$$\frac{d^3w}{dx^3} = \frac{1}{k} \cdot [2 \cdot \beta \cdot M_0 \cdot \zeta(\beta x) - T_0 \cdot \psi(\beta x)]$$



Dijagram funkcija φ=φ(β·**x**) i ψ=ψ(β·**x**)
-0.2
0.0
+0.2
+0.4
+0.6
+0.8
1.0
2.0
3.0
4.0
5.0
ψ
φ
βx
$\beta \cdot l = l \cdot \sqrt[4]{\frac{3 \cdot (1 - v^2)}{a^2 \cdot h^2}}$
β·l ≥ 5 duga ljuska
β·l < 5 kratka ljuska

Duga cilindrična ljuska opterećena unutrašnjim pritiskom Z = -p



$$w = -\frac{1}{2 \cdot \beta^3 \cdot k} \cdot [\beta \cdot M_0 \cdot \psi + T_0 \cdot \theta] - \frac{p \cdot a^2}{E \cdot h}$$

$$\frac{dw}{dx} = \frac{1}{2 \cdot \beta^2 \cdot k} \cdot [2 \cdot \beta \cdot M_0 \cdot \theta + T_0 \cdot \varphi]$$

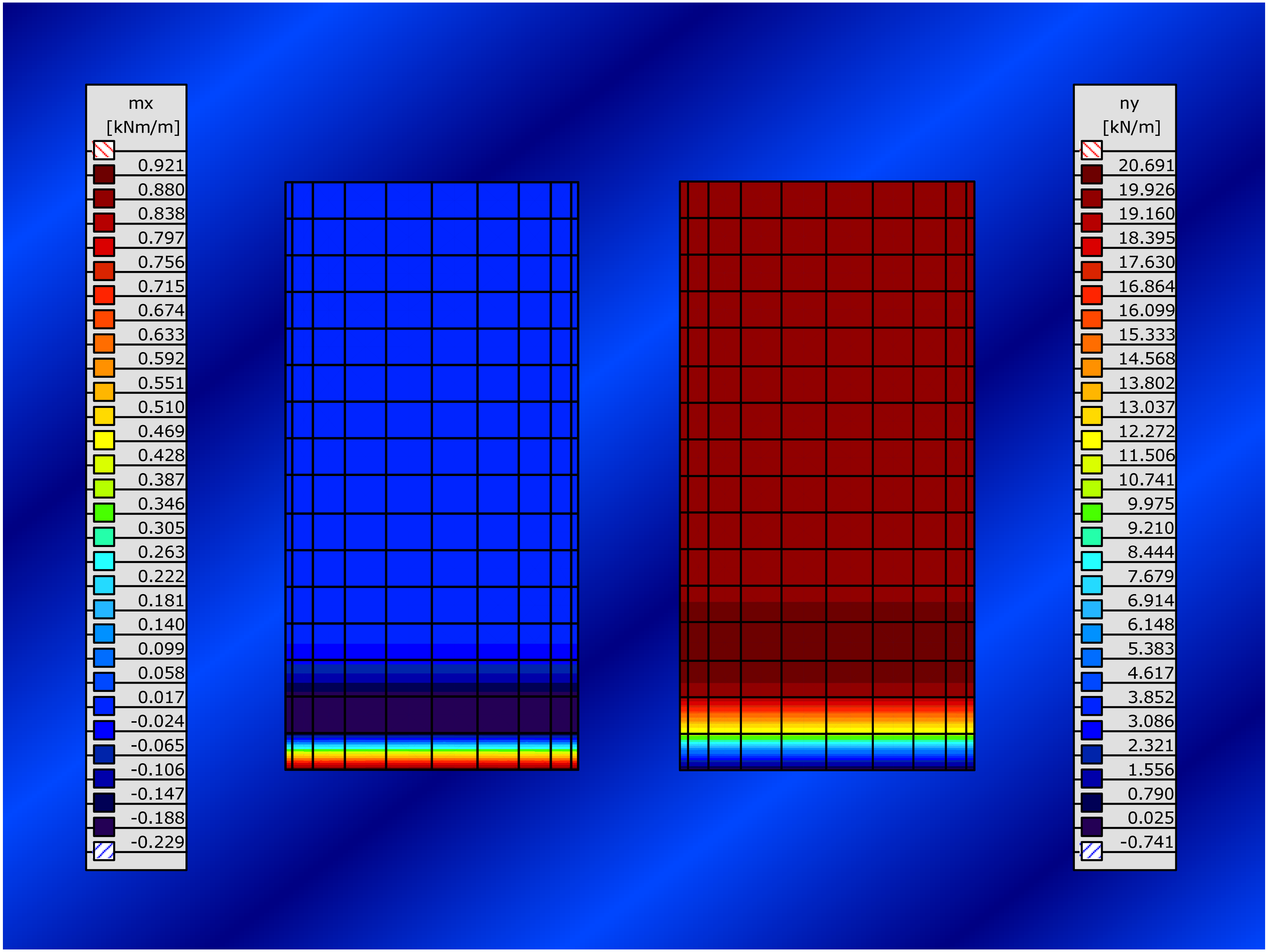
$$x = 0 \qquad w = \frac{dw}{dx} = 0$$

$$-\frac{1}{2\cdot\beta^3\cdot k}\cdot[\beta\cdot M_0+T_0]=\frac{p\cdot a^2}{E\cdot h}$$

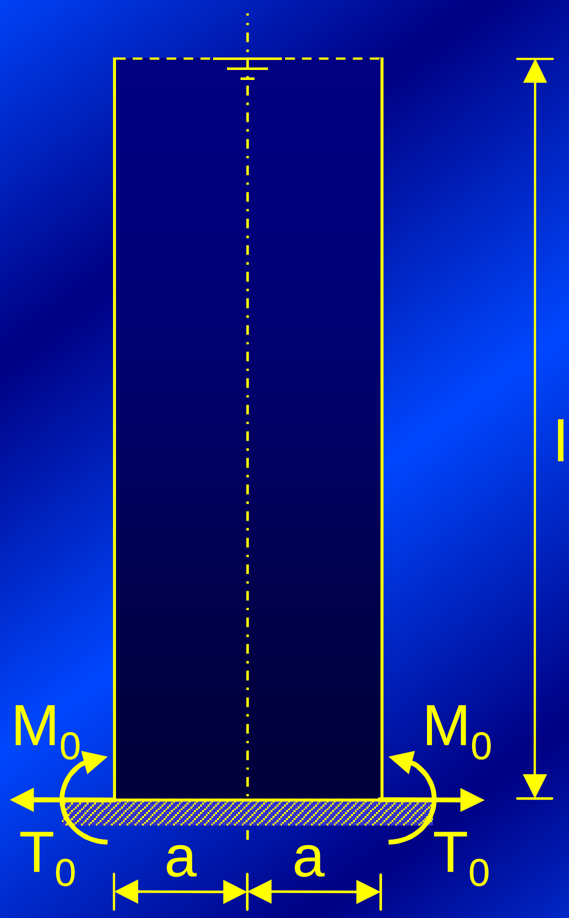
$$\frac{1}{2\cdot\beta^2\cdot k}\cdot[2\cdot\beta\cdot M_0+T_0]=0$$

$$M_0=\frac{p}{\beta^2}\qquad T_0=-\frac{2\cdot p}{\beta}$$

$$w=-p\cdot\left[\frac{\psi-2\cdot\theta}{2\cdot\beta^4\cdot k}+\frac{a^2}{E\cdot h}\right]$$


mx
[kNm/m]
0.921
0.880
0.838
0.797
0.756
0.715
0.674
0.633
0.592
0.551
0.510
0.469
0.428
0.387
0.346
0.305
0.263
0.222
0.181
0.140
0.099
0.058
0.017
-0.024
-0.065
-0.106
-0.147
-0.188
-0.229
ny
[kN/m]
20.691
19.926
19.160
18.395
17.630
16.864
16.099
15.333
14.568
13.802
13.037
12.272
11.506
10.741
9.975
9.210
8.444
7.679
6.914
6.148
5.383
4.617
3.852
3.086
2.321
1.556
0.790
0.025
-0.741

Cilindrični rezervoar ispunjen tečnošću



$$Z = -\gamma \cdot (l - x) \qquad w_0 = -\frac{\gamma \cdot (l - x) \cdot a^2}{E \cdot h}$$

$$w = -\frac{1}{2 \cdot \beta^3 \cdot k} \cdot [\beta \cdot M_0 \cdot \psi + T_0 \cdot \theta] - \frac{p \cdot a^2}{E \cdot h}$$

$$x = 0 \qquad w = 0 \qquad \frac{dw}{dx} = 0$$

$$-\frac{1}{2\cdot\beta^3\cdot k}\cdot[\beta\cdot M_0+T_0]=\frac{\gamma\cdot a^2}{E\cdot h}\cdot l$$

$$\frac{1}{2\cdot\beta^2\cdot k}\cdot[2\cdot\beta\cdot M_0+T_0]=\frac{\gamma\cdot a^2}{E\cdot h}$$

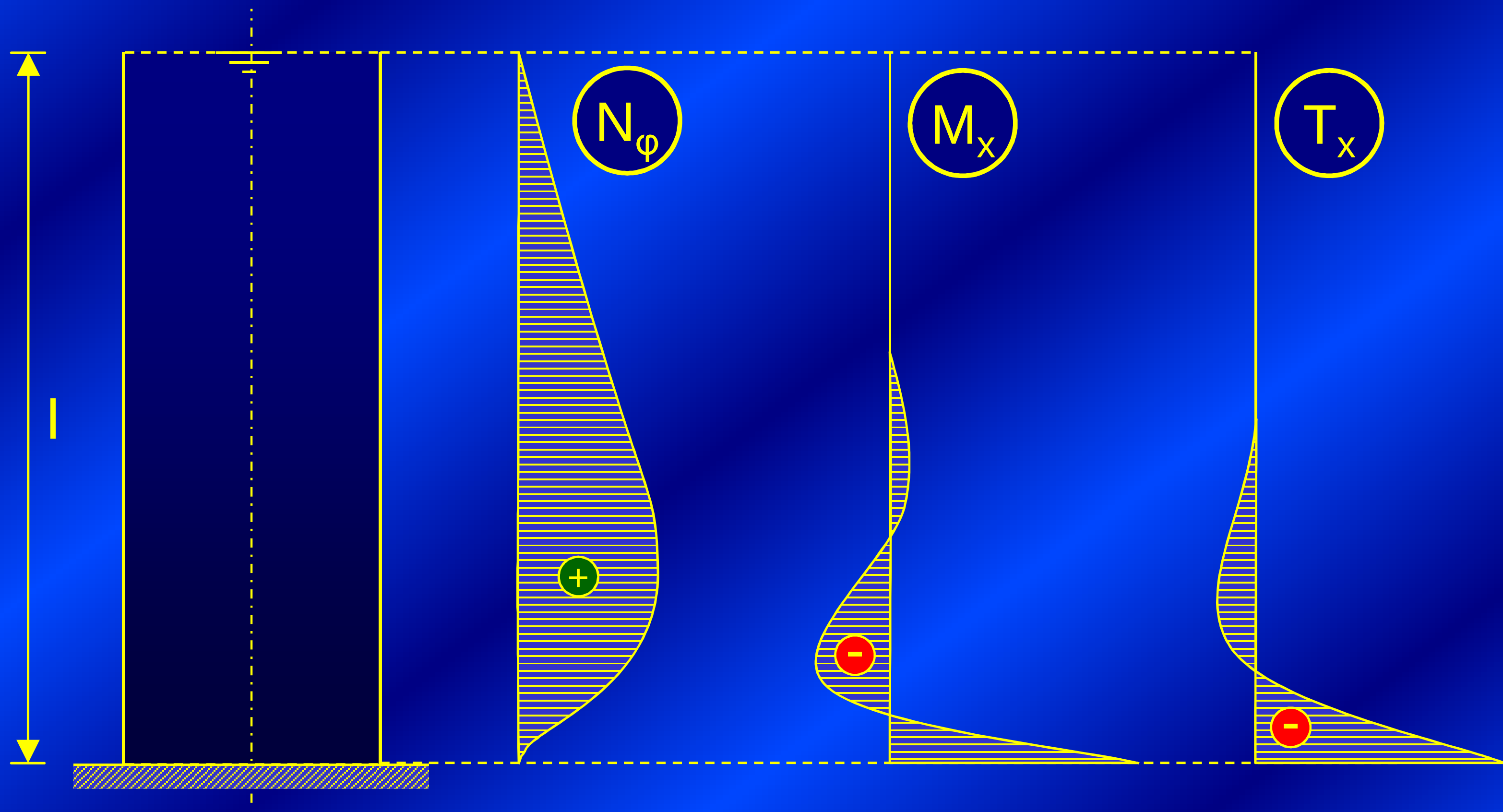
$$M_0=\frac{2\cdot\gamma\cdot a^2}{E\cdot h}\cdot\beta^2\cdot k\cdot\left(1-\frac{1}{\beta\cdot l}\right)$$

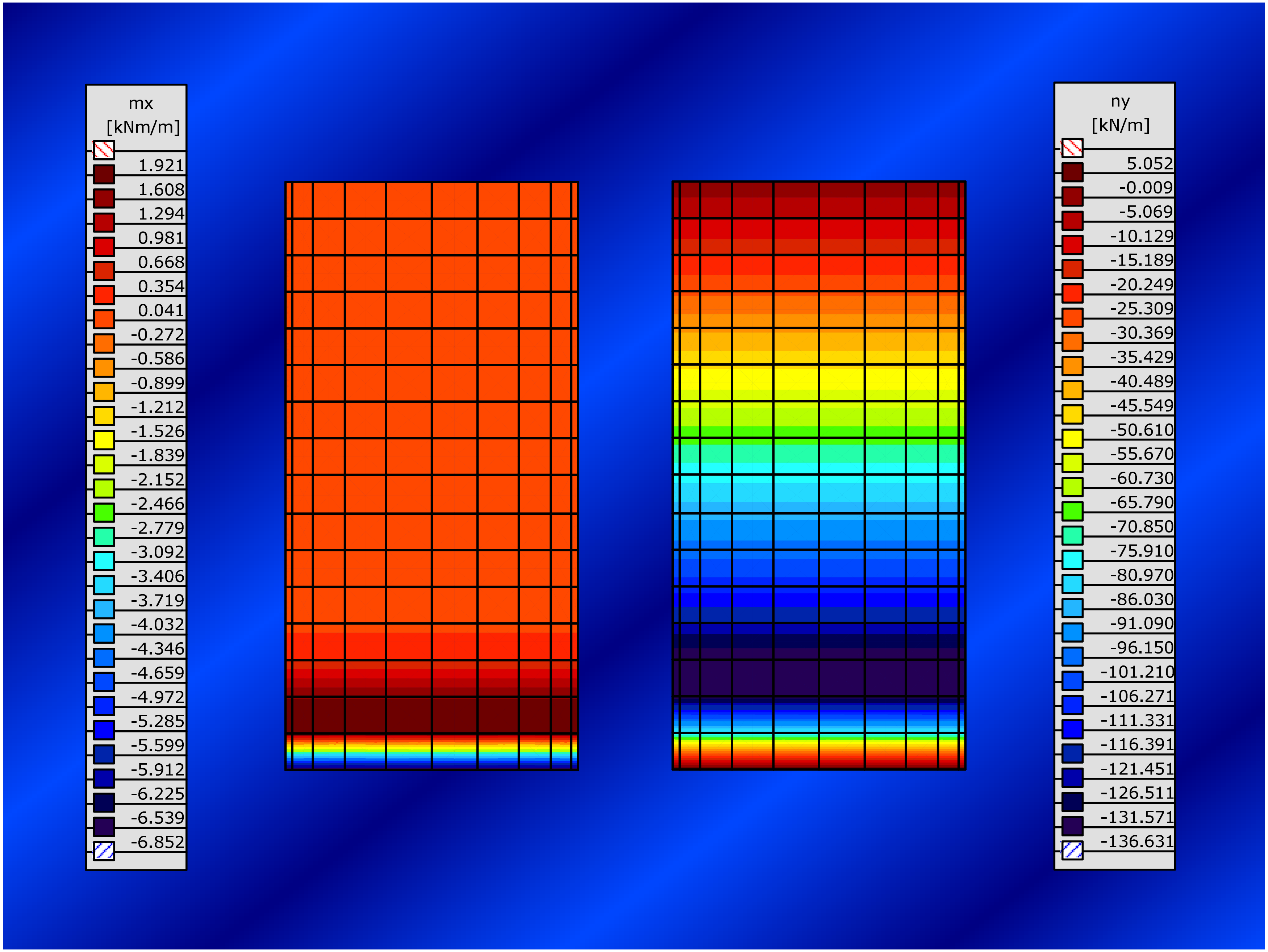
$$T_0=\frac{2\cdot\gamma\cdot a^2}{E\cdot h}\cdot\beta^3\cdot k\cdot\left(\frac{1}{\beta\cdot l}-2\right)$$

$$w=-\frac{\gamma\cdot a^2\cdot l}{E\cdot h}\cdot\left[1-\frac{x}{l}-\varphi(\beta x)+\frac{1}{\beta\cdot l}\cdot\zeta(\beta x)\right]$$

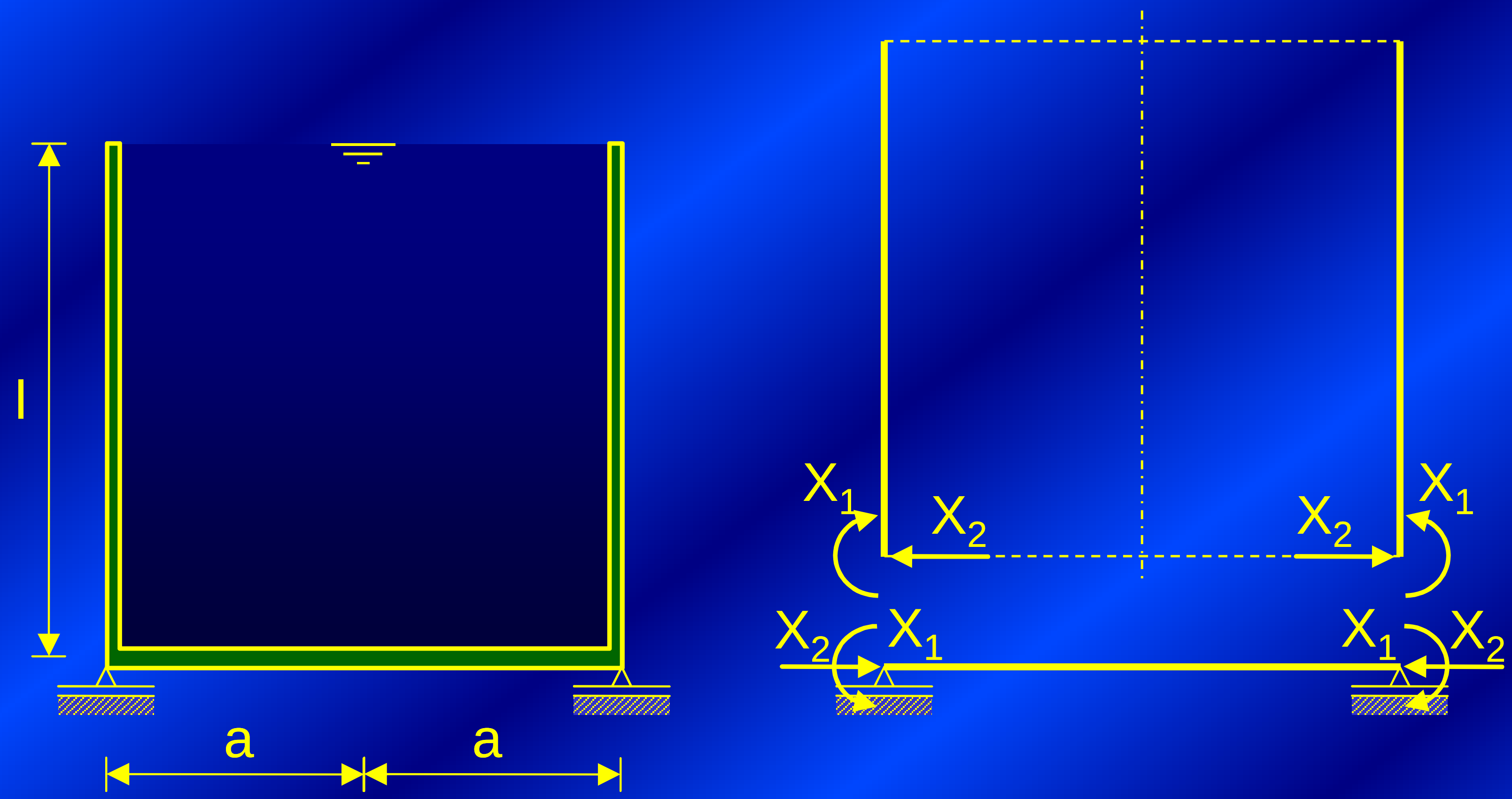
$$N_\varphi = \gamma \cdot a \cdot l \cdot \left[1 - \frac{x}{l} - \varphi(\beta x) + \frac{1}{\beta \cdot l} \cdot \zeta(\beta x) \right]$$

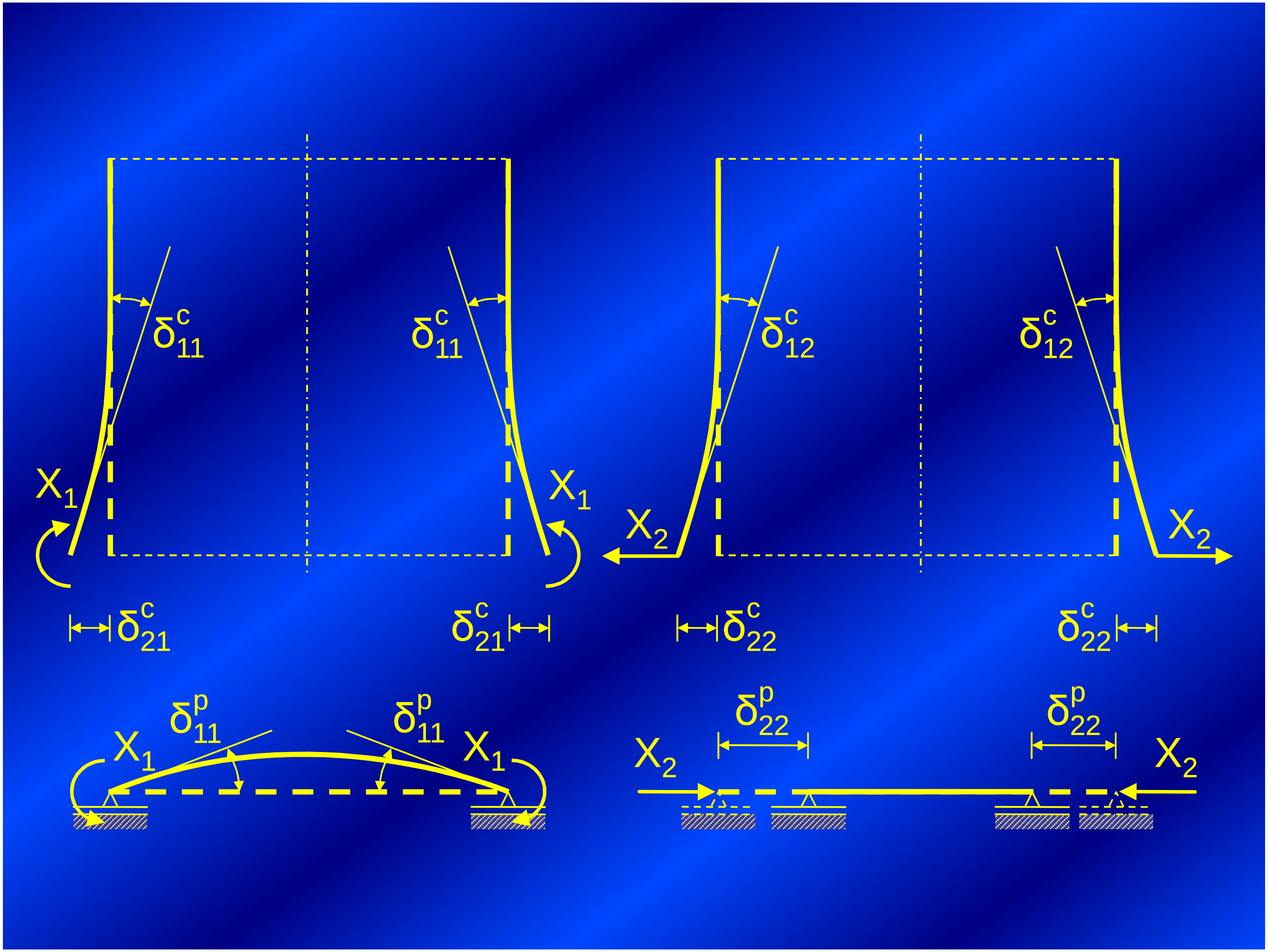
$$M_x = -k \cdot \frac{d^2 w}{dx^2} = \frac{2 \cdot \beta^2 \cdot \gamma \cdot a^2 \cdot k \cdot l}{E \cdot h} \cdot \left[1 - \frac{1}{\beta \cdot l} \cdot \theta(\beta x) - \zeta(\beta x) \right]$$



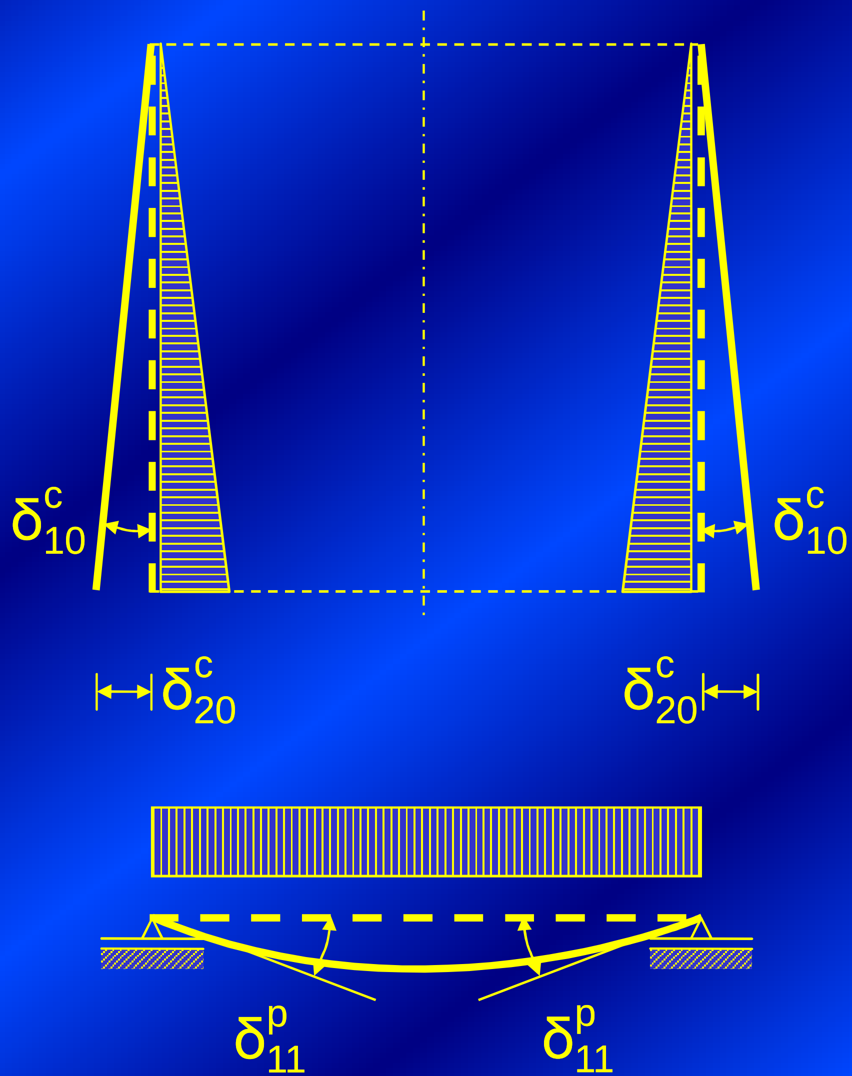

mx
[kNm/m]
1.921
1.608
1.294
0.981
0.668
0.354
0.041
-0.272
-0.586
-0.899
-1.212
-1.526
-1.839
-2.152
-2.466
-2.779
-3.092
-3.406
-3.719
-4.032
-4.346
-4.659
-4.972
-5.285
-5.599
-5.912
-6.225
-6.539
-6.852
ny
[kN/m]
5.052
-0.009
-5.069
-10.129
-15.189
-20.249
-25.309
-30.369
-35.429
-40.489
-45.549
-50.610
-55.670
-60.730
-65.790
-70.850
-75.910
-80.970
-86.030
-91.090
-96.150
-101.210
-106.271
-111.331
-116.391
-121.451
-126.511
-131.571
-136.631

Cilindrični rezervoar sa kružnom pločom





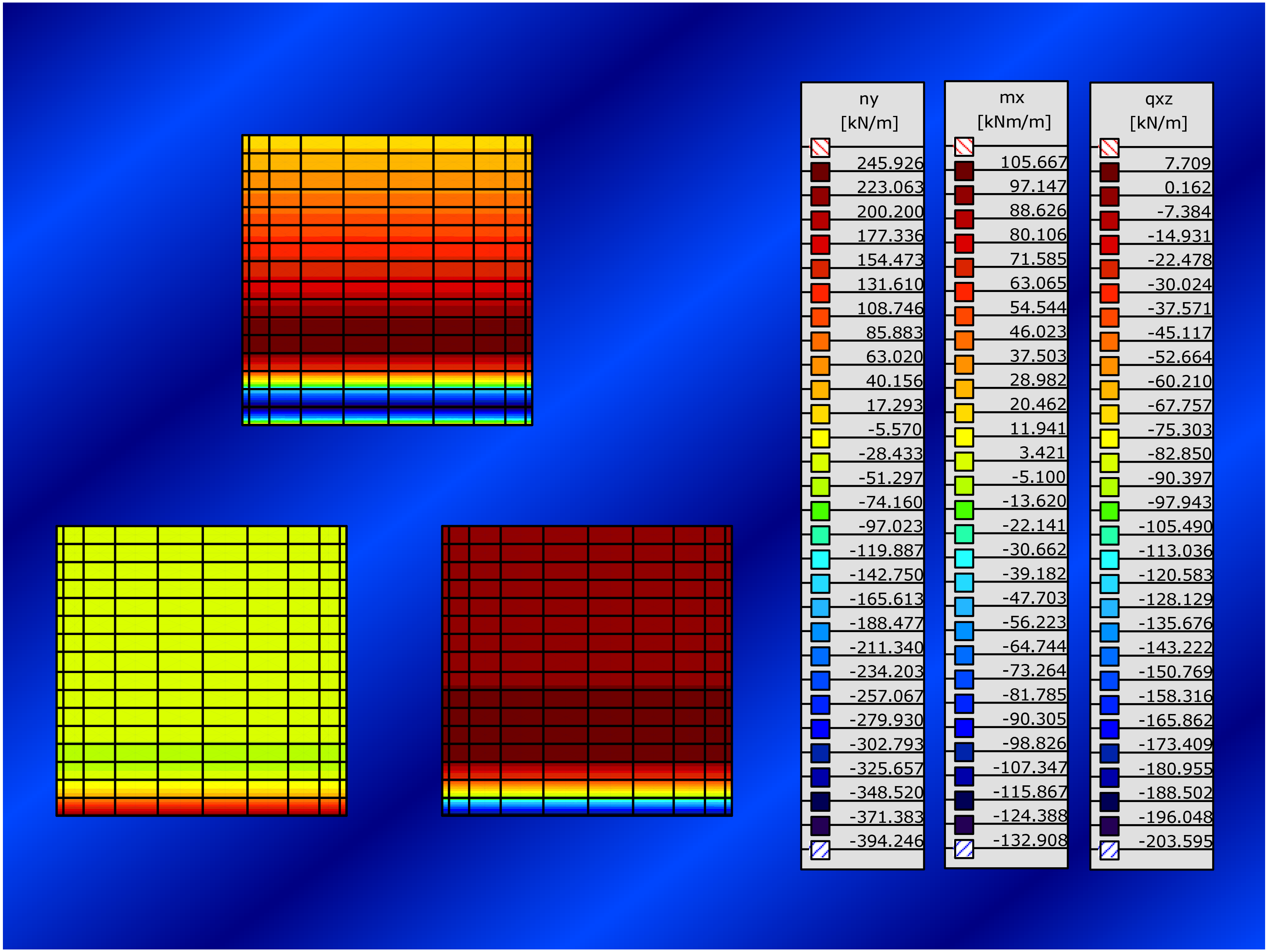
δ^c_11
δ^c_11
δ^c_12
δ^c_12
X1
X1
X2
X2
δ^c_21
δ^c_21
δ^c_22
δ^c_22
δ^p_11
δ^p_11
X1
X1
δ^p_22
δ^p_22
X2
X2



$$\delta_{11} = \delta_{11}^{c} + \delta_{11}^{p}$$
$$\delta_{12} = \delta_{12}^{c} + \delta_{12}^{p}$$
$$\delta_{21} = \delta_{21}^{c} + \delta_{21}^{p}$$
$$\delta_{22} = \delta_{22}^{c} + \delta_{22}^{p}$$
$$\vdots$$
$$\delta_{ik} = \delta_{ik}^{c} + \delta_{ik}^{p}$$

$$\delta_{11} \cdot X_1 + \delta_{12} \cdot X_2 + \delta_{10} = 0$$
$$\delta_{21} \cdot X_1 + \delta_{22} \cdot X_2 + \delta_{20} = 0$$

$$Z = Z_0 + Z_1 \cdot X_1 + Z_2 \cdot X_2$$


ny
[kN/m]
245.926
223.063
200.200
177.336
154.473
131.610
108.746
85.883
63.020
40.156
17.293
-5.570
-28.433
-51.297
-74.160
-97.023
-119.887
-142.750
-165.613
-188.477
-211.340
-234.203
-257.067
-279.930
-302.793
-325.657
-348.520
-371.383
-394.246
mx
[kNm/m]
105.667
97.147
88.626
80.106
71.585
63.065
54.544
46.023
37.503
28.982
20.462
11.941
3.421
-5.100
-13.620
-22.141
-30.662
-39.182
-47.703
-56.223
-64.744
-73.264
-81.785
-90.305
-98.826
-107.347
-115.867
-124.388
-132.908
qxz
[kN/m]
7.709
0.162
-7.384
-14.931
-22.478
-30.024
-37.571
-45.117
-52.664
-60.210
-67.757
-75.303
-82.850
-90.397
-97.943
-105.490
-113.036
-120.583
-128.129
-135.676
-143.222
-150.769
-158.316
-165.862
-173.409
-180.955
-188.502
-196.048
-203.595

Kratka cilindrična ljuska

$$\frac{d^4w}{dx^4} + 4 \cdot \beta^4 \cdot w = 0$$

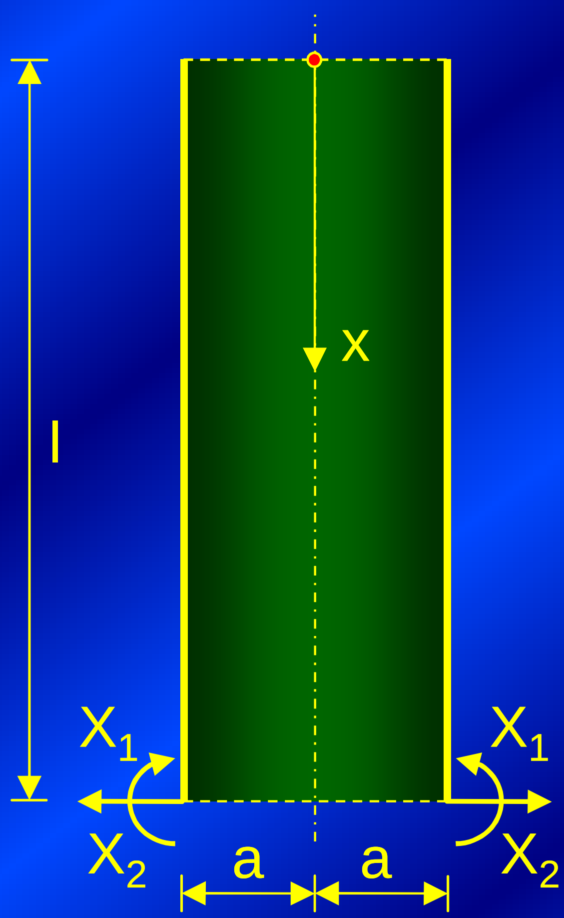
$$w = \overline{C}_1 \cdot ch\beta x \cdot \cos\beta x + \overline{C}_2 \cdot sh\beta x \cdot \sin\beta x +$$

$$+ \overline{C}_3 \cdot sh\beta x \cdot \cos\beta x + \overline{C}_4 \cdot ch\beta x \cdot \sin\beta x)$$

$$\varphi_1(\alpha) = ch\alpha \cdot \cos\alpha \qquad \varphi_2(\alpha) = \frac{ch\alpha \cdot \sin\alpha + sh\alpha \cdot \cos\alpha}{2}$$

$$\varphi_3(\alpha) = \frac{sh\alpha \cdot \sin\alpha}{2} \qquad \varphi_4(\alpha) = \frac{ch\alpha \cdot \sin\alpha - sh\alpha \cdot \cos\alpha}{4}$$

$$w = C_1 \cdot \varphi_1(\beta x) + C_2 \cdot \varphi_2(\beta x) + C_3 \cdot \varphi_3(\beta x) + C_4 \cdot \varphi_4(\beta x)$$



m	φ^{I}_{m}	φ^{II}_{m}	φ^{III}_{m}	φ^{IV}_{m}
1	$-4 \cdot \varphi_4$	$-4 \cdot \varphi_3$	$-4 \cdot \varphi_2$	$-4 \cdot \varphi_1$
2	φ_1	$-4 \cdot \varphi_4$	$-4 \cdot \varphi_3$	$-4 \cdot \varphi_2$
3	φ_2	φ_1	$-4 \cdot \varphi_4$	$-4 \cdot \varphi_3$
4	φ_3	φ_2	φ_1	$-4 \cdot \varphi_4$

$$x = 0 \qquad \frac{d^2 w}{dx^2} = 0 \qquad \frac{d^3 w}{dx^3} = 0$$

$$C_3 = C_4 = 0 \qquad w = C_1 \cdot \varphi_1(\beta x) + C_2 \cdot \varphi_2(\beta x)$$

$$x = l \qquad \frac{d^2w}{dx^2} = -\frac{X_1}{k} \qquad \frac{d^3w}{dx^3} = -\frac{X_2}{k}$$

$$\frac{1}{\beta^2}\cdot\left(\frac{d^2w}{dx^2}\right)_{x=l} = 4\cdot C_1\cdot\varphi_3(\beta l) - 4C_2\cdot\varphi_4(\beta l) = -\frac{X_1}{k\cdot\beta^2}$$

$$\frac{1}{\beta^3}\cdot\left(\frac{d^3w}{dx^3}\right)_{x=l} = 4\cdot C_1\cdot\varphi_2(\beta l) - 4C_2\cdot\varphi_3(\beta l) = -\frac{X_2}{k\cdot\beta^3}$$

$$C_1 = \frac{1}{4\cdot k\cdot\beta^3}\cdot\frac{1}{\Delta}\cdot[\beta\cdot X_1\cdot\varphi_3(\beta l) - X_2\cdot\varphi_4(\beta l)]$$

$$C_2 = \frac{1}{4\cdot k\cdot\beta^3}\cdot\frac{1}{\Delta}\cdot[X_2\cdot\varphi_3(\beta l) - \beta\cdot X_1\cdot\varphi_2(\beta l)]$$

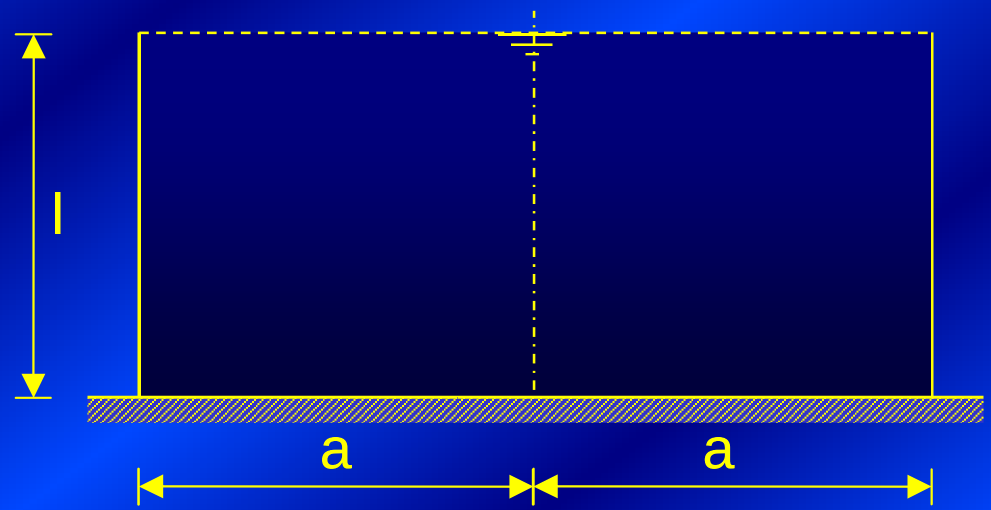
$$\Delta = \varphi_3^2(\beta l) - \varphi_2(\beta l)\cdot\varphi_4(\beta l) = \frac{sh^2\alpha - \sin^2\alpha}{8}$$

$$w = \frac{2}{k\cdot\beta^2\cdot(sh^2\alpha - \sin^2\alpha)}\cdot$$

$$\cdot\{X_1\cdot\beta\cdot[\varphi_3(\beta l)\cdot\varphi_1(\beta x) - \varphi_2(\beta l)\cdot\varphi_2(\beta x)] +$$

$$+ X_2\cdot[\varphi_3(\beta l)\cdot\varphi_2(\beta x) - \varphi_4(\beta l)\cdot\varphi_1(\beta x)]\}$$

Kratka cilindrična ljuska
- rezervoar ispunjen tečnošću



$$x = 0 \qquad \frac{d^2w}{dx^2} = 0 \qquad \frac{d^3w}{dx^3} = 0$$

$$C_3 = C_4 = 0 \qquad w = C_1 \cdot \varphi_1(\beta x) + C_2 \cdot \varphi_2(\beta x) - \frac{\gamma \cdot a^2}{E \cdot h} \cdot x$$

$$x = l \qquad w = \frac{dw}{dx} = 0$$

$$C_1 \cdot \varphi_1(\beta l) - C_2 \cdot \varphi_2(\beta l) = \frac{\gamma \cdot a^2}{E \cdot h} \cdot l$$

$$-4 \cdot C_1 \cdot \varphi_4(\beta l) - C_2 \cdot \varphi_1(\beta l) = \frac{\gamma \cdot a^2}{E \cdot h \cdot \beta}$$

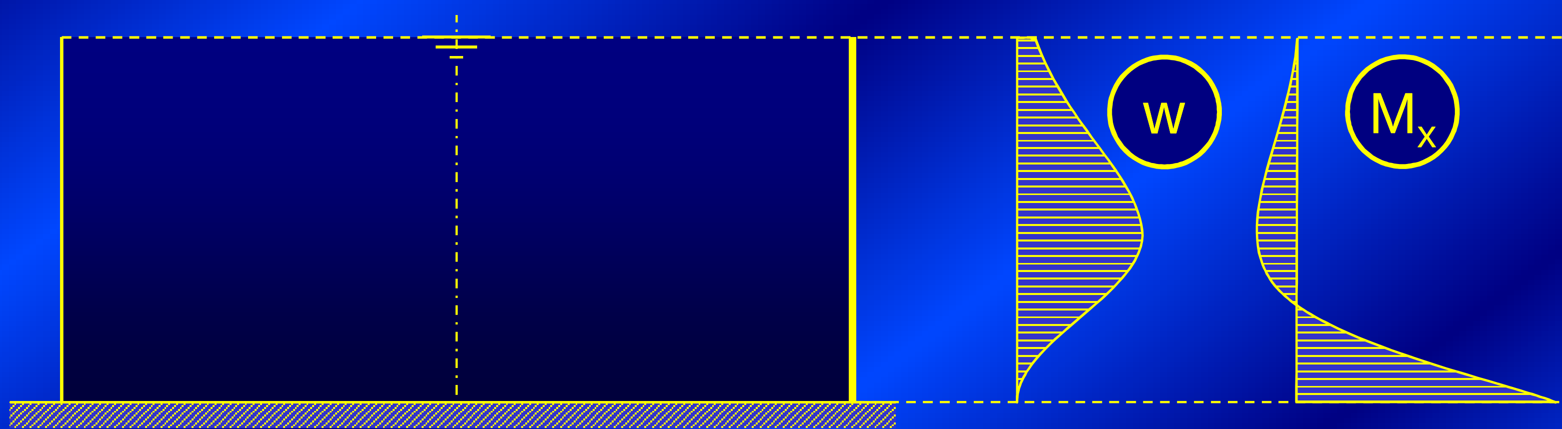
$$C_1 = \frac{1}{\Delta} \cdot \frac{\gamma \cdot a^2 \cdot l}{E \cdot h} \cdot \left[\varphi_1(\beta l) - \frac{\varphi_2(\beta l)}{\beta \cdot l} \right]$$

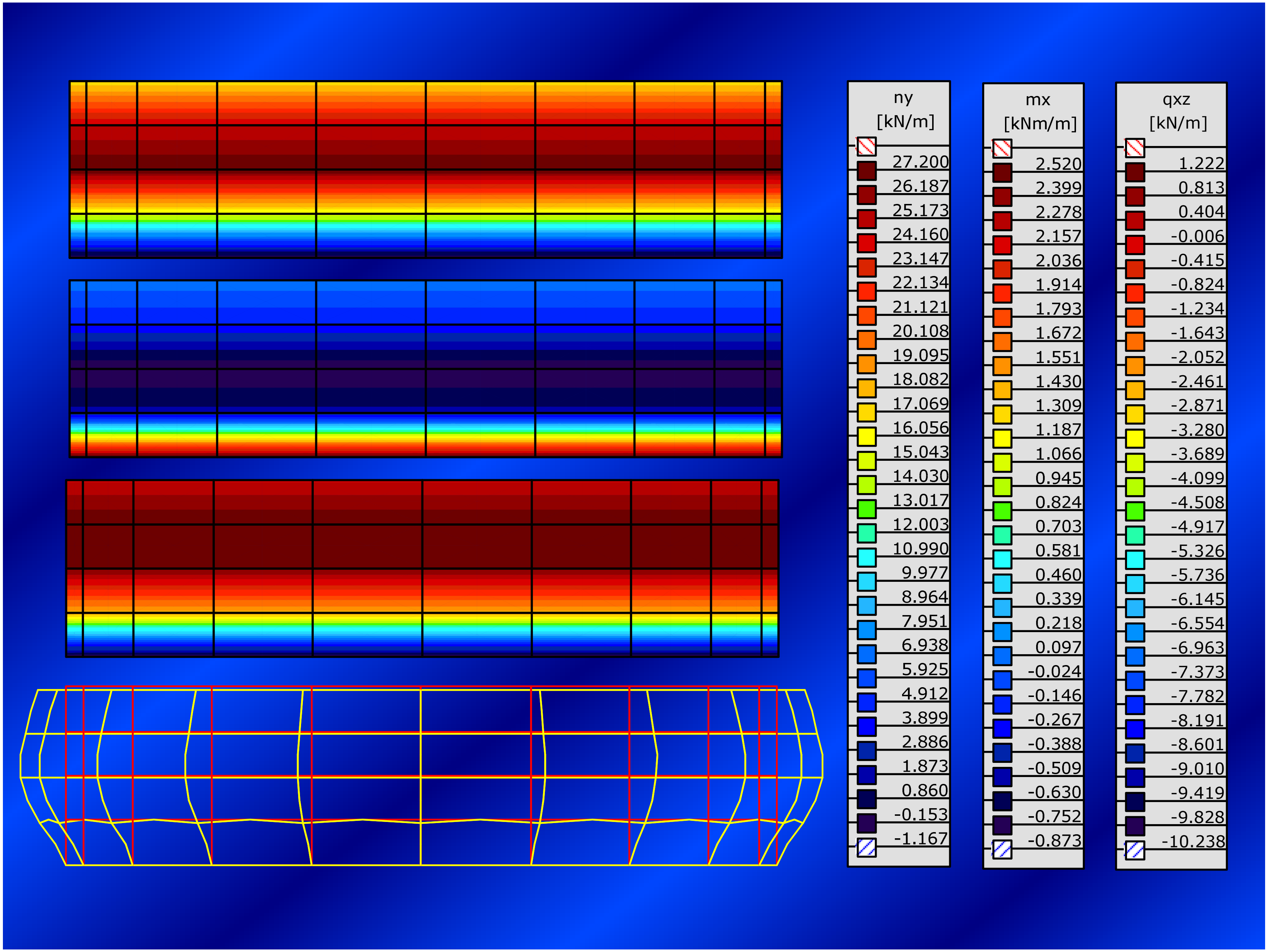
$$C_2 = \frac{1}{\Delta} \cdot \frac{\gamma \cdot a^2 \cdot l}{E \cdot h} \cdot \frac{\varphi_1(\beta l)}{\beta \cdot l} \qquad \Delta = \varphi_1^2(\beta l) + 4 \cdot \varphi_2(\beta l) \cdot \varphi_4(\beta l)$$

$$w = -\frac{\gamma \cdot a^2 \cdot l}{E \cdot h} \cdot \left[\frac{x}{l} - \overline{C}_1 \cdot \varphi_1(\beta x) - \overline{C}_2 \cdot \varphi_2(\beta x) \right]$$

$$\overline{C}_1 = \frac{E \cdot h}{\gamma \cdot a^2 \cdot l} \cdot C_1 \qquad \overline{C}_2 = \frac{E \cdot h}{\gamma \cdot a^2 \cdot l} \cdot C_2$$

$$M_x = -\frac{\gamma \cdot a^2 \cdot h^2 \cdot \beta^2 \cdot l}{3 \cdot (1 - v^2)} \cdot \left[\overline{C}_1 \cdot \varphi_3(\beta x) - \overline{C}_2 \cdot \varphi_4(\beta x) \right]$$




ny
[kN/m]
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26.187
25.173
24.160
23.147
22.134
21.121
20.108
19.095
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16.056
15.043
14.030
13.017
12.003
10.990
9.977
8.964
7.951
6.938
5.925
4.912
3.899
2.886
1.873
0.860
-0.153
-1.167
mx
[kNm/m]
2.520
2.399
2.278
2.157
2.036
1.914
1.793
1.672
1.551
1.430
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1.066
0.945
0.824
0.703
0.581
0.460
0.339
0.218
0.097
-0.024
-0.146
-0.267
-0.388
-0.509
-0.630
-0.752
-0.873
qxz
[kN/m]
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0.404
-0.006
-0.415
-0.824
-1.234
-1.643
-2.052
-2.461
-2.871
-3.280
-3.689
-4.099
-4.508
-4.917
-5.326
-5.736
-6.145
-6.554
-6.963
-7.373
-7.782
-8.191
-8.601
-9.010
-9.419
-9.828
-10.238

Fleksiona teorija rotacionih ljuski
- rotaciona simetrija -

$X = 0 \qquad Y = Y(\varphi) \qquad Z = Z(\varphi)$

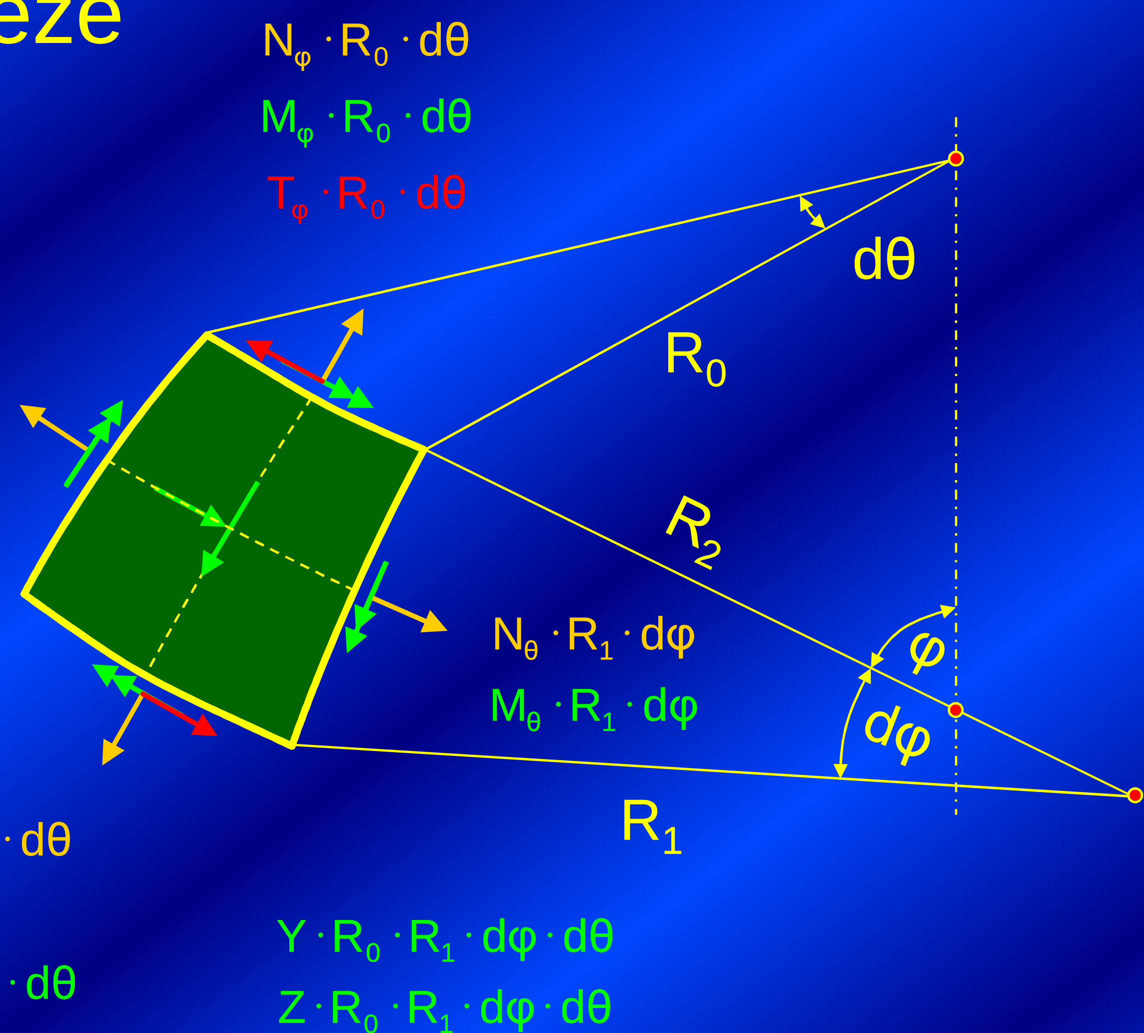
$N_\theta \neq 0 \qquad M_\theta \neq 0$

$N_\varphi \neq 0 \qquad M_\varphi \neq 0 \qquad T_\varphi \neq 0$

$N_{\varphi\theta} = 0 \qquad M_{\varphi\theta} = 0 \qquad T_\theta = 0$

$u = 0 \qquad v \neq 0 \qquad w \neq 0$

Uslovi ravnoteže



$$\left(N_\varphi \cdot R_0 + \frac{d(N_\varphi \cdot R_0)}{d\varphi} \cdot d\varphi \right) \cdot d\theta$$

$$\left(M_\varphi \cdot R_0 + \frac{d(M_\varphi \cdot R_0)}{d\varphi} \cdot d\varphi \right) \cdot d\theta$$

$$\left(T_\varphi \cdot R_0 + \frac{d(T_\varphi \cdot R_0)}{d\varphi} \cdot d\varphi \right) \cdot d\theta$$

$$\sum F_{\varphi} = 0 \quad \Rightarrow$$

$$\frac{d(N_{\varphi} \cdot R_0)}{d\varphi} - N_{\theta} \cdot R_1 \cdot \cos\varphi - T_{\varphi} \cdot R_0 + Y \cdot R_0 \cdot R_1 = 0$$

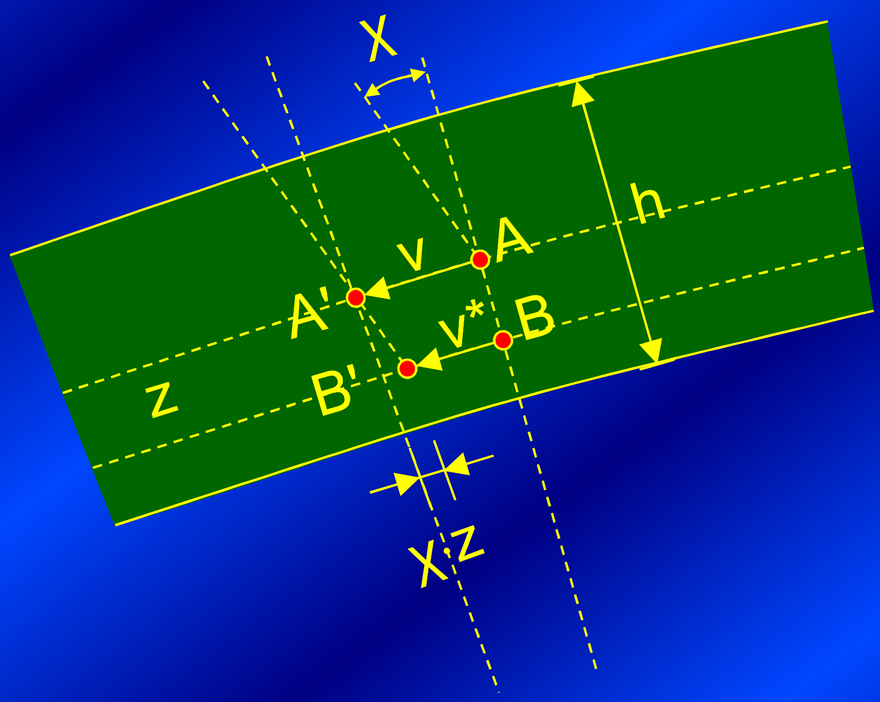
$$\sum F_{n} = 0 \quad \Rightarrow$$

$$N_{\varphi} \cdot R_0 - N_{\theta} \cdot R_1 \cdot \sin\varphi - \frac{d(T_{\varphi} \cdot R_0)}{d\varphi} + Z \cdot R_0 \cdot R_1 = 0$$

$$\sum M_{\theta} = 0 \quad \Rightarrow \quad \frac{d(M_{\varphi} \cdot R_0)}{d\varphi} - M_{\theta} \cdot R_1 \cdot \cos\varphi - T_{\varphi} \cdot R_0 \cdot R_1 = 0$$

Deformacija ljuske

$$\varepsilon_\varphi = \frac{1}{R_1}\cdot\left(\frac{dv}{d\varphi} - w\right) \qquad \varepsilon_\theta = \frac{1}{R_2}\cdot(v\cdot tg\varphi - w) \qquad \chi = \frac{1}{R_1}\cdot\left(v + \frac{dw}{d\varphi}\right)$$



$$v^* = v - \chi\cdot z$$

$$\varepsilon_\varphi^* = \frac{1}{R_1}\cdot\left(\frac{dv^*}{d\varphi} - w^*\right)$$

$$\varepsilon_\theta^* = \frac{1}{R_2}\cdot(v^*\cdot tg\varphi - w^*)$$

$$\varepsilon_\varphi{}^* = \frac{1}{R_1}\cdot\left(\frac{dv}{d\varphi} - w - \frac{d\chi}{d\varphi}\cdot z\right)$$

$$\varepsilon_\theta{}^* = \frac{1}{R_2}\cdot\left(v\cdot tg\varphi - w - \chi\cdot z\cdot ctg\varphi\right)$$

$$R_1 \to R_1' \qquad R_2 = \frac{R_0}{\sin\varphi} \to R_2'$$

$$R_1' = \frac{R_1\cdot(1+\varepsilon_\varphi)\cdot d\varphi}{d(\varphi+\chi)}$$

$$R_2' = \frac{R_0 + dR_0}{\sin(\varphi+\chi)}$$

$$\kappa_\varphi = \frac{1}{R_1'} - \frac{1}{R_1} = \frac{d(\varphi - \chi)}{R_1 \cdot d\varphi \cdot (1 + \varepsilon_\varphi)} - \frac{1}{R_1}$$

$$\kappa_\theta = \frac{1}{R_2'} - \frac{1}{R_2} = \frac{\sin(\varphi - \chi)}{R_0 + dR_0} - \frac{1}{R_2}$$

$$\varepsilon_\varphi << 1 \qquad \frac{dR_0}{R_0} << 1 \qquad \sin\chi \approx \chi \qquad \cos\varphi \approx 1$$

$$\kappa_\varphi = \frac{d(\varphi - \chi)}{R_1 \cdot d\varphi} - \frac{1}{R_1} = \frac{d\chi}{R_1 \cdot d\varphi} \qquad \kappa_\theta = \frac{\sin\varphi - \chi \cdot \cos\varphi}{R_2 \cdot \sin\varphi} - \frac{1}{R_2} = \frac{\chi}{R_2} \cdot \mathrm{ctg}\varphi$$

$$\varepsilon_\varphi{}^* = \varepsilon_\varphi - z \cdot \kappa_\varphi \qquad \varepsilon_\theta{}^* = \varepsilon_\theta - z \cdot \kappa_\theta$$

Veze deformacija, napona i sila u presecima

$$\sigma_\varphi = \frac{E}{1-v^2}\cdot(\varepsilon_\varphi * - v\cdot\varepsilon_\theta *)$$

$$\sigma_\theta = \frac{E}{1-v^2}\cdot(\varepsilon_\theta * - v\cdot\varepsilon_\varphi *)$$

$$N_\varphi = \frac{E\cdot h}{1-v^2}\cdot(\varepsilon_\varphi - v\cdot\varepsilon_\theta)$$

$$N_\theta = \frac{E\cdot h}{1-v^2}\cdot(\varepsilon_\theta - v\cdot\varepsilon_\varphi)$$

$$N_{\varphi} = \frac{E \cdot h}{1 - v^2} \cdot \left[\frac{1}{R_1} \cdot \left(\frac{dv}{d\varphi} - w \right) + \frac{v}{R_2} \cdot (v \cdot ctg\varphi - w) \right]$$

$$N_{\theta} = \frac{E \cdot h}{1 - v^2} \cdot \left[\frac{1}{R_2} \cdot (v \cdot ctg\varphi - w) + \frac{v}{R_1} \cdot \left(\frac{dv}{d\varphi} - w \right) \right]$$

$$M_{\varphi} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{\varphi} \cdot z \cdot dz = -k \cdot (\kappa_{\varphi} + v \cdot \kappa_{\theta})$$

$$M_{\theta} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{\theta} \cdot z \cdot dz = -k \cdot (\kappa_{\theta} + v \cdot \kappa_{\varphi})$$

$$k = \frac{E \cdot h^3}{12 \cdot (1 - v^2)}$$

$$M_\varphi = -k \cdot \left(\frac{d\chi}{R_1 \cdot d\varphi} + v \cdot \frac{\chi}{R_2} \cdot ctg\varphi \right)$$

$$M_\theta = -k \cdot \left(\frac{\chi}{R_2} \cdot ctg\varphi + v \cdot \frac{d\chi}{R_1 \cdot d\varphi} \right)$$

Sferna kupola opterećena silama na konturi

$$X = Y = Z = 0 \qquad R_1 = R_2 = a \qquad R_0 = a \cdot \sin\varphi$$

$$\frac{d(N_\varphi \cdot \sin\varphi)}{d\varphi} - T_\varphi \cdot \sin\varphi - N_\theta \cdot \cos\theta = 0$$

$$(N_\varphi + N_\theta) \cdot \sin\varphi + \frac{d(T \cdot \sin\varphi)}{d\varphi} = 0$$

$$\frac{d}{d\varphi}(M_\varphi \cdot \sin\varphi) - M_\theta \cdot \cos\varphi - T \cdot a \cdot \sin\varphi = 0$$

$$\frac{d^2\chi}{d\varphi^2} + \frac{d\chi}{d\varphi} \cdot \mathrm{ctg}\varphi - \chi \cdot (v + \mathrm{ctg}^2\varphi) + \frac{a^2}{k} \cdot T = 0$$

$$N_\varphi = -T \cdot \mathrm{ctg}\varphi$$

$$N_\theta = -\frac{dT}{d\varphi}$$

$$(\varepsilon_\varphi - \varepsilon_\theta) \cdot \mathrm{ctg}\varphi - \frac{d\varepsilon_\theta}{d\varphi} = \chi$$

$$\frac{d^2T}{d\varphi^2} + \frac{dT}{d\varphi} \cdot \mathrm{ctg}\varphi + T \cdot (v - \mathrm{ctg}^2\varphi) = E \cdot h \cdot \chi$$

Sferna ljuska opterećena silama na konturi - približno rešenje

$$\varphi \uparrow \quad \Rightarrow \quad \chi \ll \frac{d\chi}{d\varphi} \qquad \frac{d\chi}{d\varphi} \ll \frac{d^2\chi}{d\varphi^2}$$

$$\frac{d^2\chi}{d\varphi^2} + \frac{a^2}{k} \cdot T = 0 \qquad \frac{d^2T}{d\varphi^2} = E \cdot h \cdot \chi$$

$$\frac{d^4\chi}{d\varphi^4} + E \cdot h \cdot \frac{a^2}{k} \cdot \chi = 0 \qquad \frac{d^4T}{d\varphi^4} + E \cdot h \cdot \frac{a^2}{k} \cdot T = 0$$

$$\aleph = \sqrt{\frac{a}{h} \cdot \sqrt{3 \cdot (1 - \nu^2)}}$$

$$\frac{d^4T}{d\varphi^4} + 4 \cdot \aleph^4 \cdot T = 0 \qquad T = e^{r \cdot \varphi}$$

$$r^4 + 4 \cdot \aleph^4 = 0$$

$$r_{1/2} = \pm(1+i) \cdot \aleph \qquad r_{3/4} = \pm(1-i) \cdot \aleph$$

$$T = e^{\aleph \cdot \varphi} \cdot (C_1 \cdot \cos \aleph\varphi + C_2 \cdot \sin \aleph\varphi) +$$

$$+ e^{-\aleph \cdot \varphi} \cdot (C_3 \cdot \cos \aleph\varphi + C_4 \cdot \sin \aleph\varphi)$$

$$T = e^{\aleph \cdot \varphi} \cdot (C_1 \cdot \cos \aleph\varphi + C_2 \cdot \sin \aleph\varphi)$$

$$\omega = \alpha - \varphi \qquad T = Ce^{-\aleph \cdot \varphi} \cdot \cos(\aleph\varphi + \psi)$$

$$C_1 = C \cdot e^{-\aleph \cdot \alpha} \cdot \cos(\aleph\varphi + \psi)$$

$$C_2 = C \cdot e^{-\aleph \cdot \alpha} \cdot \sin(\aleph\varphi + \psi)$$

$$\frac{dT}{d\varphi} = \aleph \cdot \sqrt{2} \cdot C \cdot e^{-\aleph \cdot \omega} \cdot \sin\left(\aleph\omega + \psi + \frac{\pi}{2}\right)$$

$$\frac{d^2T}{d\varphi^2} = 2 \cdot \aleph^2 \cdot C \cdot e^{-\aleph \cdot \omega} \cdot \sin(\aleph\omega + \psi)$$

$$\chi = 2 \cdot \frac{\aleph}{E \cdot h} \cdot C \cdot e^{-\aleph \cdot \omega} \cdot \sin(\aleph\omega + \psi)$$

$$\frac{d\chi}{d\varphi} = 2 \cdot \frac{\aleph^3 \cdot \sqrt{2}}{E \cdot h} \cdot C \cdot e^{-\aleph \cdot \omega} \cdot \cos\left(\aleph\omega + \psi + \frac{\pi}{4}\right)$$

$$N_\varphi = -C \cdot e^{-\aleph \cdot \omega} \cdot \mathrm{ctg}\varphi \cdot \cos(\aleph\omega + \psi)$$

$$N_\theta = \aleph \cdot \sqrt{2} \cdot C \cdot e^{-\aleph \cdot \omega} \cdot \sin\left(\aleph\omega + \psi + \frac{\pi}{4}\right)$$

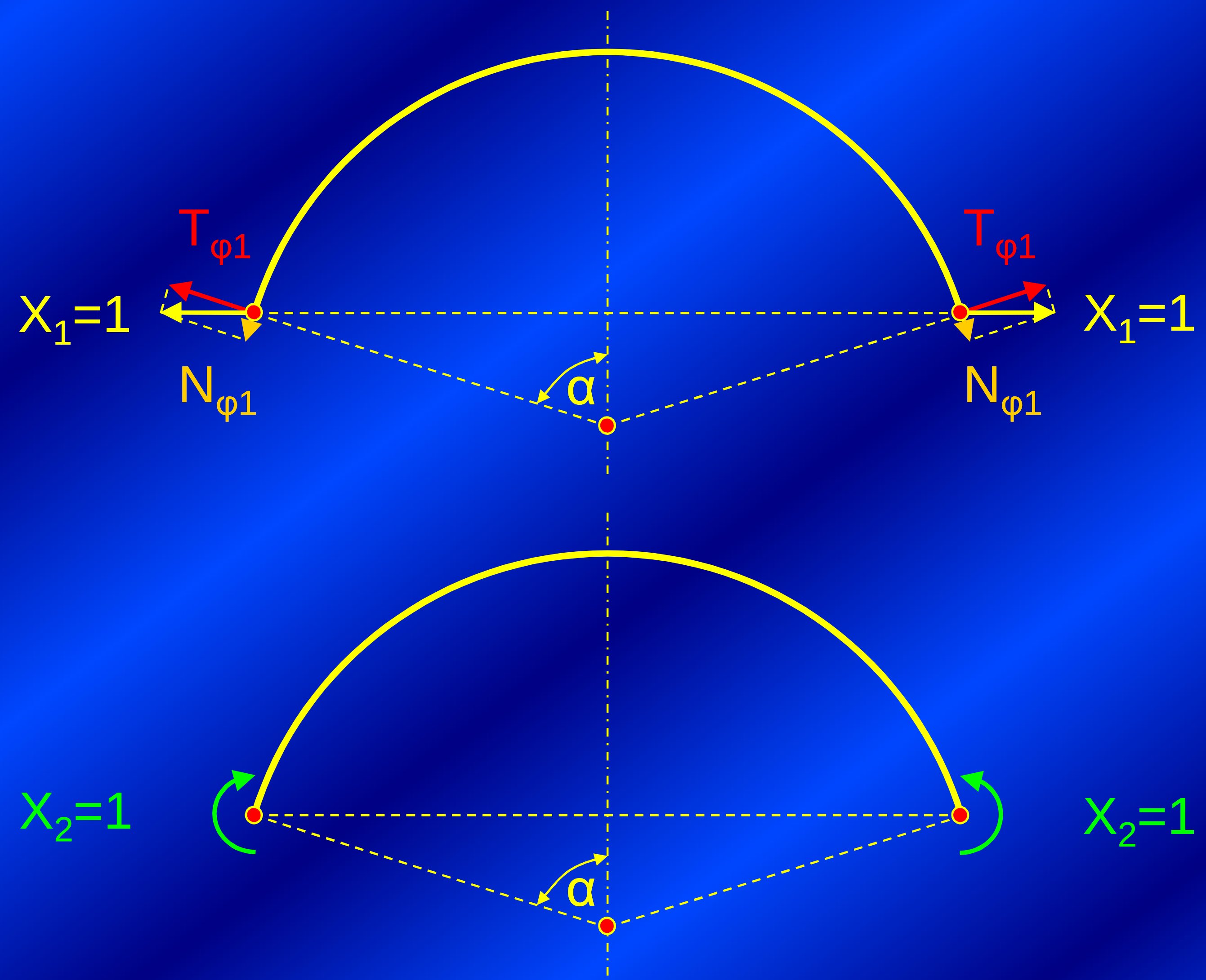
$$M_\varphi = -\frac{k}{a}\cdot\frac{d\chi}{d\varphi} = C\cdot\frac{k}{a}\cdot\frac{2\cdot\aleph^3\cdot\sqrt{2}}{E\cdot h}\cdot\cos\left(\aleph\omega+\psi+\frac{\pi}{4}\right)$$

$$M_\theta = -k\cdot\frac{\chi}{a}+v\cdot M_\varphi$$

$$M_\varphi = C\cdot\frac{a}{\aleph\cdot\sqrt{2}}\cdot e^{-\aleph\cdot\omega}\cdot\cos\left(\aleph\omega+\psi+\frac{\pi}{4}\right)$$

$$M_\theta = -\frac{C\cdot a\cdot \mathrm{ctg}\varphi}{2\cdot\aleph^2}\cdot e^{-\aleph\cdot\omega}\cdot\cos(\aleph\omega+\psi)+v\cdot M_\varphi$$

Sferna ljuska - sile na konturi



$$X_1 = 1 \qquad \varphi = \alpha$$

$$T_{\varphi 1} = -\sin\alpha$$

$$M_{\varphi 1} = 0 \quad \rightarrow \quad \frac{C \cdot a}{\aleph \cdot \sqrt{2}} \cdot \cos\left(\psi + \frac{\pi}{4}\right) = 0 \quad \Rightarrow \quad \psi = \frac{\pi}{4}$$

$$N_{\varphi 1} = \cos\alpha \quad \rightarrow \quad -C \cdot \mathrm{ctg}\alpha \cos\frac{\pi}{4} = \cos\alpha \quad \Rightarrow \quad C = -\sqrt{2} \cdot \sin\alpha$$

$$T_{\varphi 1} = -\sqrt{2} \cdot \sin\alpha \cdot e^{-\aleph \cdot \omega} \cdot \cos\left(\aleph\omega + \frac{\pi}{4}\right)$$

$$N_{\varphi 1} = \sqrt{2} \cdot \sin\alpha \cdot e^{-\aleph \cdot \omega} \cdot \mathrm{ctg}\varphi \cdot \cos\left(\aleph\omega + \frac{\pi}{4}\right)$$

$$N_{\theta 1} = 2 \cdot \aleph \cdot e^{-\aleph \cdot \omega} \cdot \sin\alpha \cdot \cos\aleph\omega$$

$$M_{\varphi 1} = \frac{a}{\aleph} \cdot e^{-\aleph \cdot \omega} \cdot \sin\alpha \cdot \cos\aleph\omega$$

$$M_{\theta 1} = -\frac{a}{\sqrt{2} \cdot \aleph^2} \cdot e^{-\aleph \cdot \omega} \cdot \sin\alpha \cdot \mathrm{ctg}\varphi \cdot \sin\left(\aleph\omega + \frac{\pi}{4}\right) + v \cdot M_{\varphi 1}$$

$$\chi_1 = -\frac{2\cdot\sqrt{2}\cdot\aleph^2}{E\cdot h}\cdot e^{-\aleph\cdot\omega}\cdot\sin\alpha\cdot\sin\left(\aleph\omega+\frac{\pi}{4}\right)$$

$$\varphi = \alpha \quad \rightarrow \quad \chi_1 = -\frac{2\cdot\sqrt{2}}{E\cdot h}\cdot\sin\alpha$$

$$\Delta R_{01} = \varepsilon_{\theta 1}\cdot R_0 = \frac{2\cdot\aleph\cdot R_0}{E\cdot h}\cdot\sin\alpha$$

$$X_2 = 1 \qquad \varphi = \alpha$$

$$T_{\varphi 2} = 0 \quad \longrightarrow \quad -C \cdot \cos\psi \quad \Rightarrow \quad \psi = \frac{\pi}{2}$$

$$N_{\varphi 2} = 0 \quad \longrightarrow \quad -C \cdot \operatorname{ctg}\alpha \cos\varphi = 0$$

$$M_{\varphi 2} = 1 \quad \longrightarrow \quad \frac{C \cdot a}{\aleph \cdot \sqrt{2}} \cdot \cos\frac{3 \cdot \pi}{4} = 1 \quad \Rightarrow \quad C = -\frac{2 \cdot \aleph}{a}$$

$$T_{\varphi 2} = \frac{2 \cdot \aleph}{a} \cdot e^{-\aleph \cdot \omega} \cdot \sin \aleph \omega$$

$$N_{\varphi 2} = -\frac{2 \cdot \aleph}{a} \cdot e^{-\aleph \cdot \omega} \cdot \mathrm{ctg}\varphi \cdot \sin \aleph \omega$$

$$N_{\theta 2} = \frac{2 \cdot \aleph^2}{a} \cdot e^{-\aleph \cdot \omega} \cdot (\cos \aleph \omega - \sin \aleph \omega)$$

$$M_{\varphi 2} = e^{-\aleph \cdot \omega} \cdot (\cos \aleph \omega + \sin \aleph \omega)$$

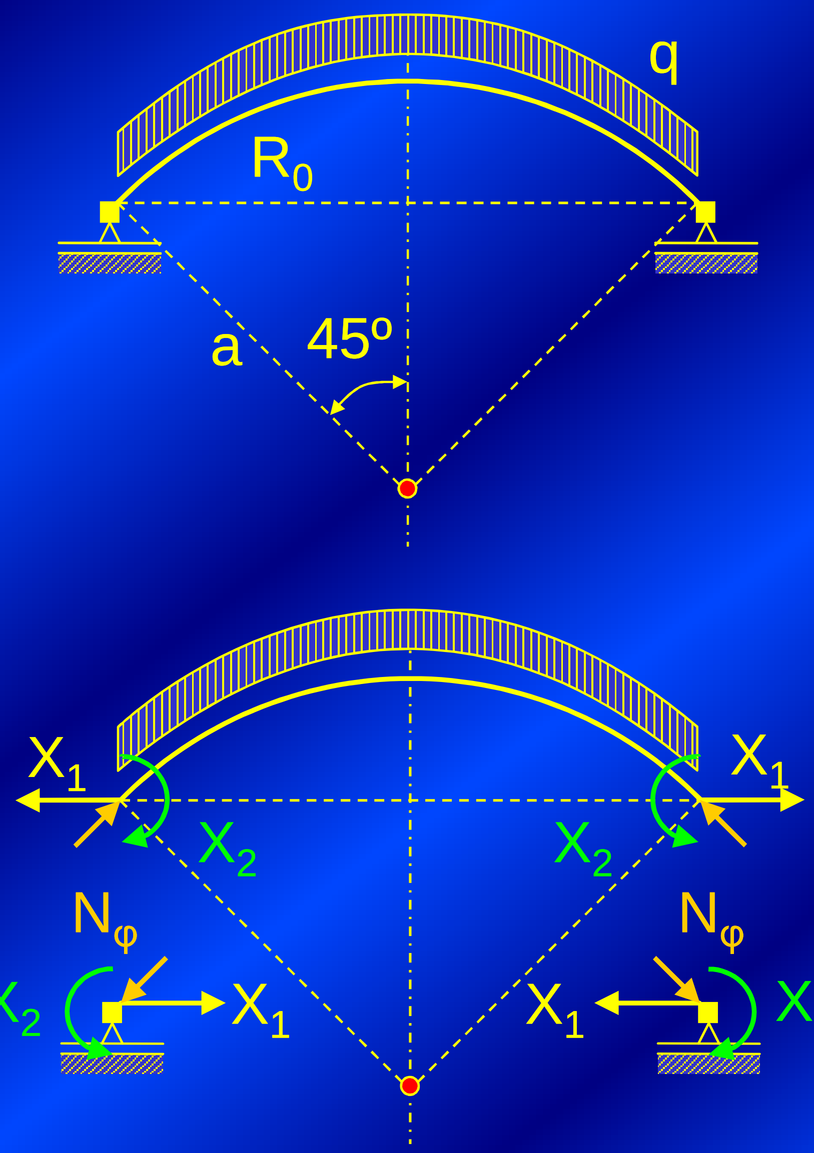
$$M_{\theta 2} = \frac{e^{-\aleph \cdot \omega}}{\aleph} \cdot \mathrm{ctg}\varphi \cdot \cos \aleph \omega + v \cdot M_{\varphi 1}$$

$$x_2 = -\frac{a}{k \cdot \aleph} \cdot e^{-\aleph \cdot \omega} \cdot \cos \aleph \omega$$

$$\varphi = \alpha \quad \rightarrow \quad x_2 = -\frac{a}{k \cdot \aleph}$$

$$\Delta R_{02} = \varepsilon_{\theta 2} \cdot R_0 = \frac{2 \cdot \aleph^2 \cdot \sin \alpha}{E \cdot h}$$

Sferna ljuska sa prstenom na konturi



δ^{I}_{ij} - pomeranje ljuske na mestu i u pravcu sile X_i usled sile X_j

δ^{II}_{ij} - pomeranje prstena na mestu i u pravcu sile X_i usled sile X_j

$$\delta_{11} \cdot X_1 + \delta_{12} \cdot X_2 + \delta_{10} = 0$$
$$\delta_{21} \cdot X_1 + \delta_{22} \cdot X_2 + \delta_{20} = 0$$

$$Z = Z_0 + Z_1 \cdot X_1 + Z_2 \cdot X_2$$

$$\delta_{11} = \delta^{I}_{11} + \delta^{II}_{11}$$
$$\delta_{12} = \delta^{I}_{12} + \delta^{II}_{12}$$
$$\delta_{21} = \delta^{I}_{21} + \delta^{II}_{21}$$
$$\delta_{22} = \delta^{I}_{22} + \delta^{II}_{22}$$
$$\vdots$$
$$\delta_{ik} = \delta^{I}_{ik} + \delta^{II}_{ik}$$

$$E \cdot \delta_{11} = E \cdot (\delta_{11}^{I} + \delta_{11}^{II}) = \frac{2 \cdot \aleph \cdot R_0}{h} \cdot \sin\alpha + \frac{4 \cdot (R_0 + b/2)^2}{b \cdot d}$$

$$E \cdot \delta_{12} = E \cdot (\delta_{12}^{I} + \delta_{12}^{II}) = \frac{2 \cdot \aleph^2}{h} \cdot \sin\alpha - \frac{6 \cdot (R_0 + b/2)^2}{b \cdot d^2}$$

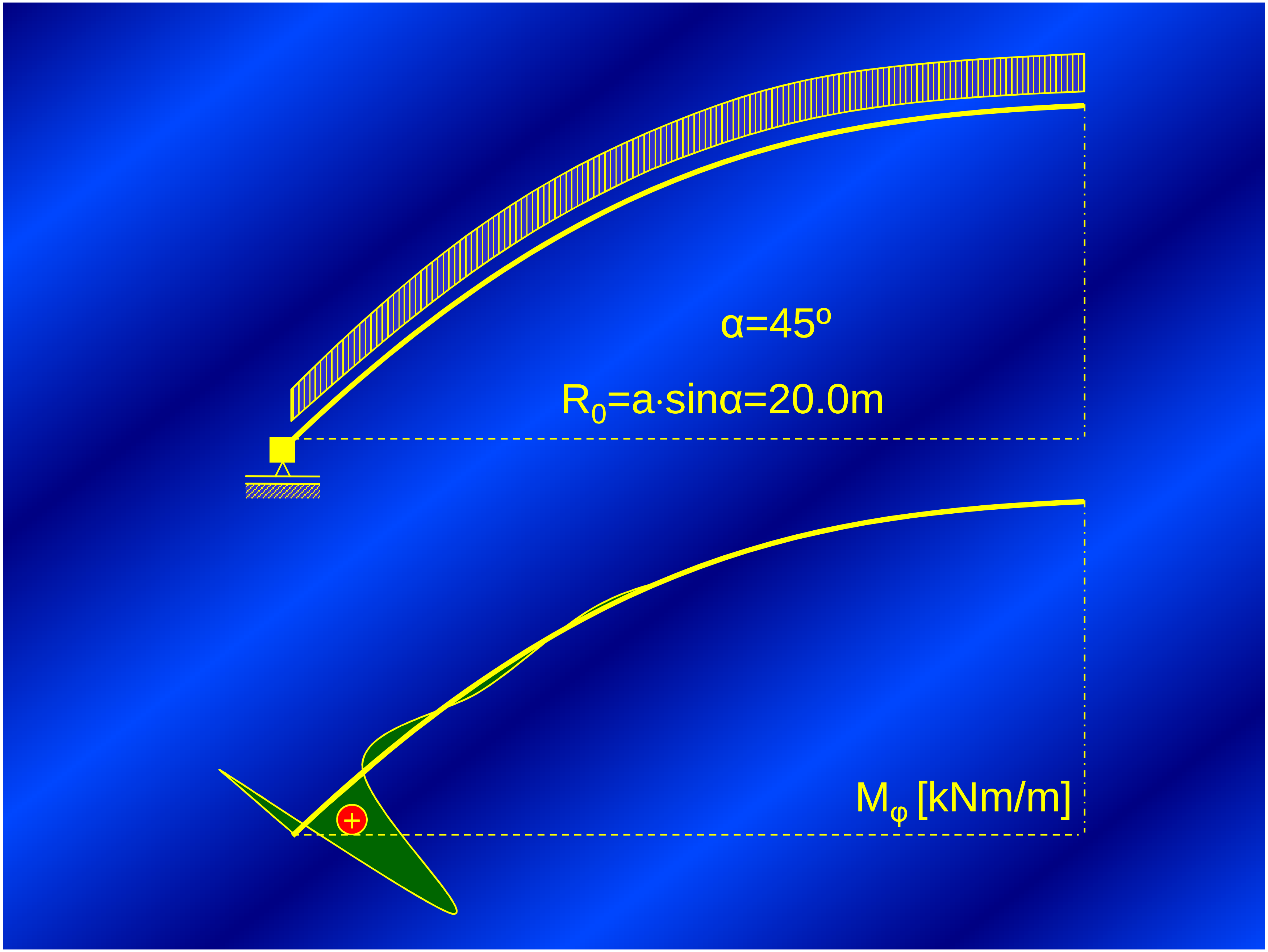
$$E \cdot \delta_{21} = E \cdot (\delta_{21}^{I} + \delta_{21}^{II}) = \frac{2 \cdot \aleph^2}{h} \cdot \sin\alpha - \frac{6 \cdot (R_0 + b/2)^2}{b \cdot d^2}$$

$$E \cdot \delta_{22} = E \cdot (\delta_{22}^{I} + \delta_{22}^{II}) = \frac{a}{k \cdot \aleph} + \frac{12 \cdot (R_0 + b/2)^2}{b \cdot d^3}$$

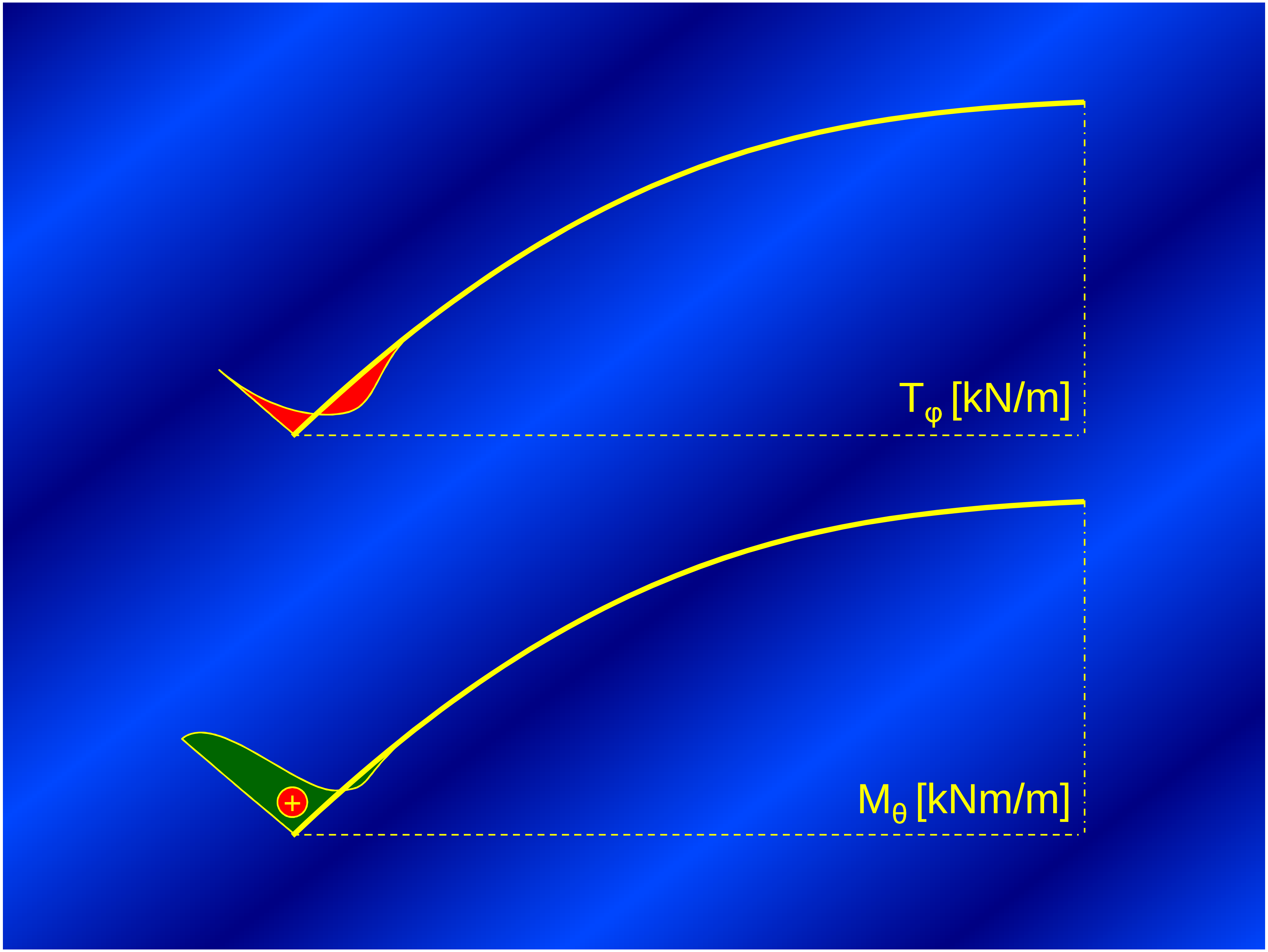
$$E\cdot\delta_{10}=E\cdot(\delta_{10}^{I}+\delta_{10}^{II})=-\frac{a^2\cdot q\cdot\sin\alpha}{h}\cdot\left(\cos\alpha-\frac{1}{1+\cos\alpha}\right)+$$

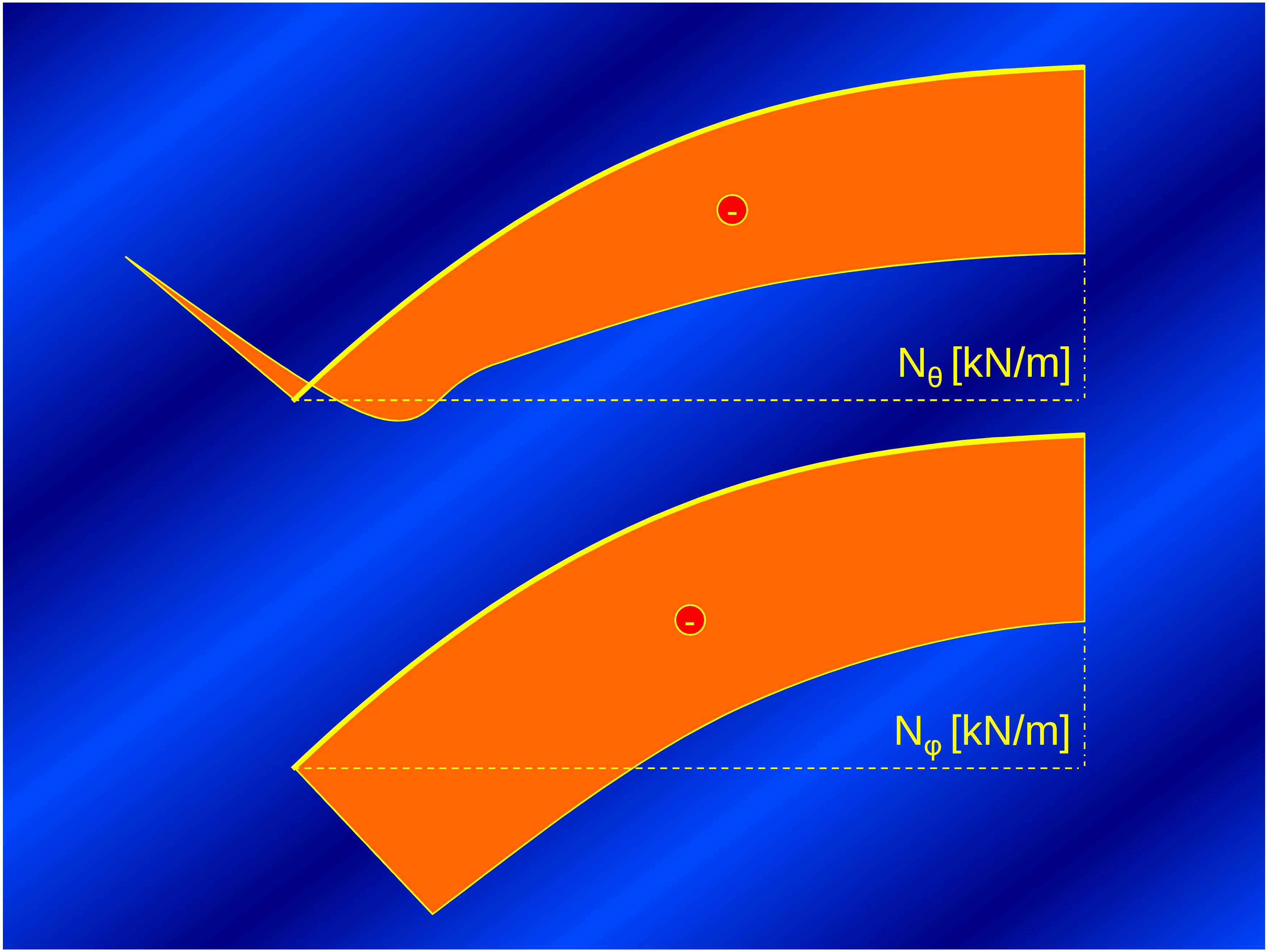
$$+\frac{N_\varphi\cdot\cos\alpha\cdot(R_0+b/2)^2}{b\cdot d}$$

$$E\cdot\delta_{20}=E\cdot(\delta_{20}^{I}+\delta_{20}^{II})=-\frac{2\cdot a\cdot q\cdot\sin\alpha}{h}+0$$

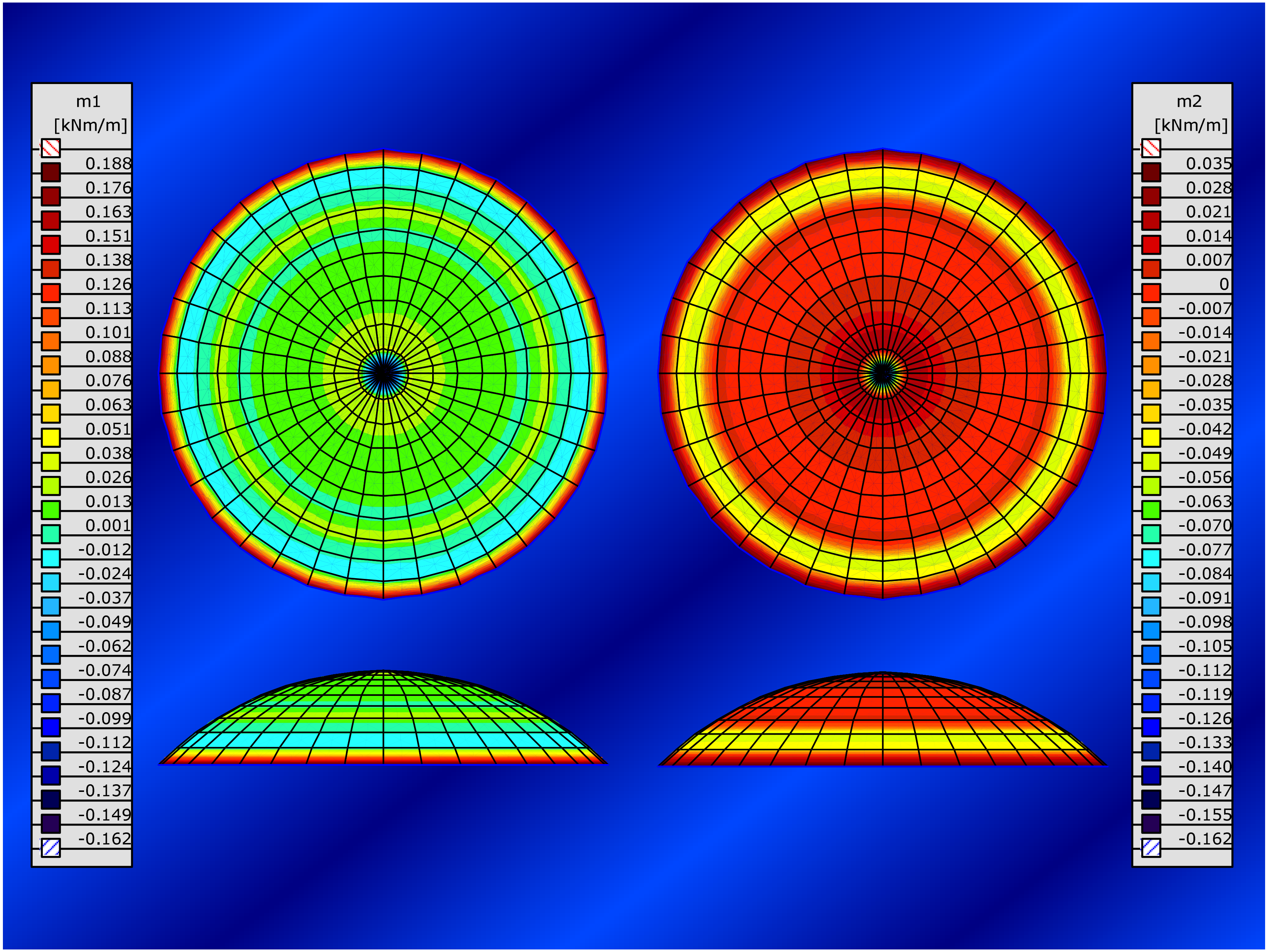


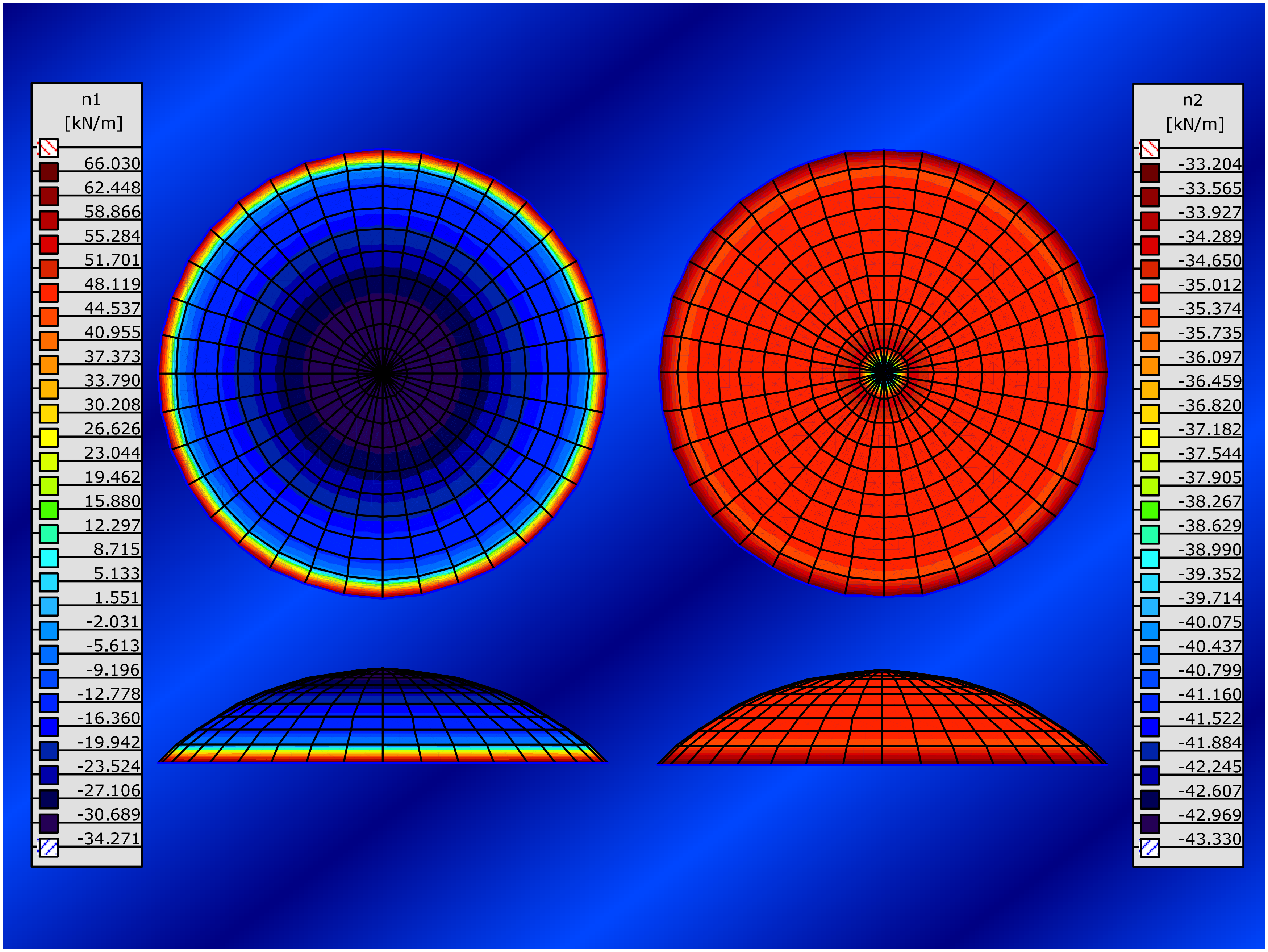
α=45º
R0=a·sinα=20.0m
+
Mφ [kNm/m]

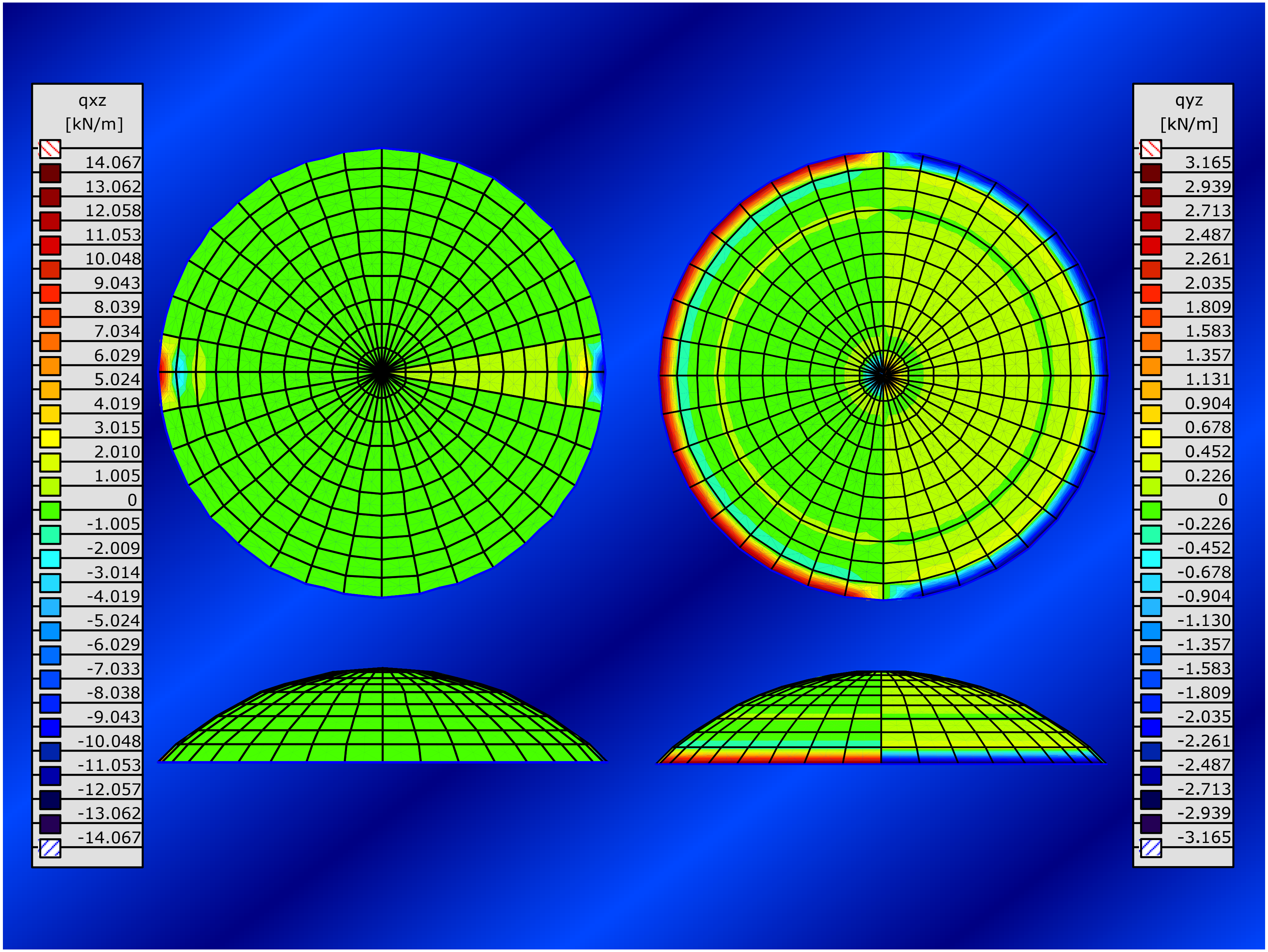

T_φ [kN/m]
M_θ [kNm/m]
+



-
N_θ [kN/m]
-
N_φ [kN/m]


m1
[kNm/m]
0.188
0.176
0.163
0.151
0.138
0.126
0.113
0.101
0.088
0.076
0.063
0.051
0.038
0.026
0.013
0.001
-0.012
-0.024
-0.037
-0.049
-0.062
-0.074
-0.087
-0.099
-0.112
-0.124
-0.137
-0.149
-0.162
m2
[kNm/m]
0.035
0.028
0.021
0.014
0.007
0
-0.007
-0.014
-0.021
-0.028
-0.035
-0.042
-0.049
-0.056
-0.063
-0.070
-0.077
-0.084
-0.091
-0.098
-0.105
-0.112
-0.119
-0.126
-0.133
-0.140
-0.147
-0.155
-0.162


n1
[kN/m]
66.030
62.448
58.866
55.284
51.701
48.119
44.537
40.955
37.373
33.790
30.208
26.626
23.044
19.462
15.880
12.297
8.715
5.133
1.551
-2.031
-5.613
-9.196
-12.778
-16.360
-19.942
-23.524
-27.106
-30.689
-34.271
n2
[kN/m]
-33.204
-33.565
-33.927
-34.289
-34.650
-35.012
-35.374
-35.735
-36.097
-36.459
-36.820
-37.182
-37.544
-37.905
-38.267
-38.629
-38.990
-39.352
-39.714
-40.075
-40.437
-40.799
-41.160
-41.522
-41.884
-42.245
-42.607
-42.969
-43.330


qxz
[kN/m]
14.067
13.062
12.058
11.053
10.048
9.043
8.039
7.034
6.029
5.024
4.019
3.015
2.010
1.005
0
-1.005
-2.009
-3.014
-4.019
-5.024
-6.029
-7.033
-8.038
-9.043
-10.048
-11.053
-12.057
-13.062
-14.067
qyz
[kN/m]
3.165
2.939
2.713
2.487
2.261
2.035
1.809
1.583
1.357
1.131
0.904
0.678
0.452
0.226
0
-0.226
-0.452
-0.678
-0.904
-1.130
-1.357
-1.583
-1.809
-2.035
-2.261
-2.487
-2.713
-2.939
-3.165